5.1: 1, 5, 9

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For all 3 problems, we solve the same equation, but apply different boundary conditions.

$y'' + \lambda y = 0$ Consider 3 case

$\lambda = 0$

$\lambda < 0 \leftarrow$ here let $\lambda = -\alpha^2$

$\lambda > 0 \leftarrow$ here let $\lambda = \alpha^2$

I. when $\lambda = 0$, $y'' = 0 \rightarrow y(t) = C_1 + C_2 t$

II. when $\lambda = -\alpha^2 < 0$

$y'' - \alpha^2 y = 0$ Guess $y = e^{rt}$

$r^2 e^{rt} - \alpha^2 e^{rt} = 0$

$r^2 = \alpha^2$

$r = \pm\alpha$

$y(t) = c_1 e^{\alpha t} + c_2 e^{-\alpha t}$

III when $\lambda = \alpha^2 > 0$

$y'' + \alpha^2 y = 0$ $y = e^{rt}$

$r^2 + \alpha^2 = 0$

$r = \pm i\alpha$

$y(t) = C_1 \cos t + C_2 \sin t$

Problem 1 $y(0) = 0$ $y'(\ell) = 0$ Try the 3 cases!

I. $y(t) = c_1 + C_2 t$ $y(0) = C_1 = 0$

$y'(t) = C_2$ $y'(\ell) = C_2 = 0$

$\Big\}$ only trivial sol^n possible

II. $y(t) = c_1 e^{\alpha t} + c_2 e^{-\alpha t}$ $\qquad$ $y(0) = c_1 + c_2 = 0$

$y'(t) = \alpha\,(c_1 e^{\alpha t} - c_2 e^{-\alpha t})$ $\qquad$ $c_2 = -c_1$

$$y'(t) = \alpha c_1 \,(e^{\alpha t} + e^{-\alpha t})$$

$$y'(\ell) = \underbrace{\alpha}_{\neq 0} c_1 \,(\underbrace{e^{\alpha \ell} + e^{-\alpha \ell}}_{\neq 0}) = 0$$

so

$c_1 = 0 \Rightarrow c_2 = 0$

only trivial solution possible.

III $y(t) = c_1 \cos \alpha t + c_2 \sin \alpha t$ $\qquad$ $y(0) = c_1 = 0$

$y'(t) = -c_1 \alpha \sin \alpha t + c_2 \alpha \cos \alpha t$

$$y'(t) = c_2 \alpha \cos \alpha t$$

$$y'(\ell) = c_2 \alpha \cos(\alpha \ell) = 0$$

either $c_2 = 0$, or $\cos(\alpha \ell) = 0$

$\Downarrow$

$\alpha = (n + \frac{1}{2})\pi$

for any integer n

Nontrivial sol^n (the eigenvalues + eigenvectors)

$$\underbrace{y(t) = \sin\left((n+\tfrac{1}{2})\pi\, t\right)}_{\text{eigenfunctions}} \quad \text{for} \quad \underbrace{\lambda = \left[(n+\tfrac{1}{2})\pi\right]^2}_{\text{eigen values}}$$

Problem 5 $\qquad$ $y(0) = 0, \quad y(\pi) - y'(\pi) = 0$

Again, consider all the cases.

I. $y(t) = c_1 + c_2 t$ $\qquad$ $y(0) = c_1 = 0$

$y'(t) = c_2$

$y(\pi) - y'(\pi) = 0$

$y(\pi) - y'(\pi) = 0$

$C_2\pi - C_2 = 0$

$C_2(\pi - 1) = 0$

$C_2 = 0$ only trivial sol^n

II. $y(t) = c_1 e^{\alpha t} + c_2 e^{-\alpha t}$

$y(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

$$
\begin{aligned}
y'(t) &= \alpha\left(c_1 e^{\alpha t} - c_2 e^{-\alpha t}\right) \\
&= \alpha\left(c_1 e^{\alpha t} + c_1 e^{-\alpha t}\right) \\
&= 2\alpha c_1 \left(\frac{e^{\alpha t} + e^{-\alpha t}}{2}\right) \\
&= 2\alpha c_1 \cosh(\alpha t)
\end{aligned}
$$

more convenient way to write this equation

$$
\begin{aligned}
y(t) &= c_1 e^{\alpha t} - c_1 e^{-\alpha t} \\
&= 2c_1\left(\frac{e^{\alpha t} - e^{-\alpha t}}{2}\right) \\
&= 2c_1 \sinh(\alpha t)
\end{aligned}
$$

more convenient way to write it

so $y(\pi) - y'(\pi) = 0$

$$2c_1 \sinh(\alpha\pi) - 2\alpha c_1 \cosh(\alpha\pi) = 0$$

$$2c_1 \underbrace{\left(\sinh(\alpha\pi) - \alpha\cosh(\alpha\pi)\right)}_{\text{nontrivial sol}^n \text{ possible if}} = 0$$

$$\sinh \alpha\pi = \alpha \cosh(\alpha\pi)$$

(you can find this value of $\alpha = \sqrt{-\lambda}$ with a computer.)

and with it, you get this sol^n

$$y(t) = c\sinh\left(\sqrt{-\lambda}\, t\right)$$

III $y(t) = c_1 \cos\alpha t + c_2 \sin\alpha t$

$y(0) = c_1 = 0$

$y(t) = c_2 \sin \alpha t$

$y'(t) = c_2 \alpha \cos \alpha t$

$y(\pi) - y'(\pi) = 0$

$c_2 \sin \alpha\pi - c_2 \alpha \cos \alpha\pi = 0$

$c_2 \underbrace{(\sin \alpha\pi - \alpha \cos \alpha\pi)} = 0$

nontrivial sol^n possible if

$\sin \alpha\pi = \alpha \cos \alpha\pi$

$\sin \alpha\pi = \alpha \cos \alpha\pi$

$\alpha = \dfrac{\sin \alpha\pi}{\cos \alpha\pi}$

$\alpha = \tan \alpha\pi$

use computers to find these α_n

$\lambda = \alpha^2 \Rightarrow \alpha_n = \sqrt{\lambda_n}$

sol is $y(t) = c_n \sin(\sqrt{\lambda_n}\, t)$

Problem 9 Finally, we consider the boundary conditions

$\underbrace{y(0) = y(2\pi)}_{\text{function begins where it ends}}, \quad \underbrace{y'(0) = y'(2\pi)}_{\text{same for its derivative}}$

I. $y(t) = c_1 + c_2 t$

$y(0) = y(2\pi)$

$c_1 = c_1 + 2\pi c_2$

$0 = 2\pi c_2$

$c_2 = 0$

$y(t) = c_1$

$y'(t) = 0 \rightarrow y'(0) = y'(2\pi)$

$0 = 0$

For $\boxed{\lambda = 0}$ any constant function $y(t) = c$

will work

II For $\lambda < 0$

$$y(t) = c_1 e^{\alpha t} + c_2 e^{-\alpha t}$$

$$y(0) = y(2\pi)$$

$$c_1 = c_1 e^{2\alpha\pi} + c_2 e^{-2\alpha\pi}$$

$$0 = c_1 (e^{2\alpha\pi} - 1) + c_2 e^{-2\alpha\pi}$$

$$y'(t) = \alpha (c_1 e^{\alpha t} - c_2 e^{-\alpha t})$$

$$y'(0) = y'(2\pi)$$

$$\alpha(c_1 + c_2) = \alpha(c_1 e^{2\pi\alpha} - c_2 e^{-2\pi\alpha})$$

alpha cancels out

$$0 = c_1(e^{2\pi\alpha} - 1) + c_2(-e^{-2\pi\alpha} - 1)$$

$$\underbrace{\begin{bmatrix} e^{2\alpha\pi} - 1 & e^{-2\alpha\pi} \\ e^{2\alpha\pi} - 1 & -e^{-2\alpha\pi} - 1 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

det(A) = 0, or only trivial sol^n

$$(e^{2\alpha\pi} - 1)(-e^{-2\alpha\pi} - 1) - (e^{2\alpha\pi} - 1)e^{-2\alpha\pi} \stackrel{?}{=} 0$$

only = 0 if $\alpha = 0$

$$-e^{-2\alpha\pi} - 1 - e^{-2\alpha\pi} \stackrel{?}{=} 0$$

$$-2e^{-2\alpha\pi} - 1 \stackrel{?}{=} 0$$

$$-2e^{-2\alpha\pi} \stackrel{?}{=} 1$$

No this isn't possible
as $e^x > 0$ for
any value of x

No sol^n for $\lambda < 0$

III $y(t) = c_1 \cos \alpha t + c_2 \sin \alpha t$

$$y(0) = y(2\pi)$$

$$c_1 = c_1 \cos 2\pi\alpha + c_2 \sin 2\pi\alpha$$

$$c_1(1 - \cos 2\pi\alpha) = c_2 \sin 2\pi\alpha$$

hmm, lets try other condition before going further

$$y'(t) = \alpha(-c_1 \sin \alpha t + c_2 \cos \alpha t)$$

$$y'(0) = y'(2\pi)$$

$$\cancel{\alpha}(c_2) = \cancel{\alpha}(-c_1 \sin 2\pi\alpha + c_2 \cos 2\pi\alpha)$$

$$c_1 \sin 2\pi\alpha = c_2(\cos 2\pi\alpha - 1)$$

$$= -c_2(1 - \cos 2\pi\alpha)$$

$$-\frac{c_1}{c_2} \sin 2\pi\alpha = (1 - \cos 2\pi\alpha)$$

$$c_1(1 - \cos 2\pi\alpha) = c_2 \sin 2\pi\alpha$$

$$(1 - \cos 2\pi\alpha) = \frac{c_2}{c_1} \sin 2\pi\alpha$$

$$-\frac{c_1}{c_2} \sin 2\pi\alpha = \frac{c_2}{c_1} \sin 2\pi\alpha$$

either $\dfrac{-c_1}{c_2} = \dfrac{c_2}{c_1}$

$\}$

$-c_1^2 = c_2^2$

both c_1, c_2 are real, so only option 2

$\Rightarrow c_1 = c_2 = 0$

or $\quad \sin 2\pi\alpha = 0$

$$2\pi\alpha = n\pi$$

so $\quad \alpha = \frac{n}{2}$

for any integer n is required

by a similar argument, however, we get:

$$\frac{c_1}{c_2}(1-\cos 2\pi\alpha) = -\frac{c_2}{c_1}(1-\cos 2\pi\alpha)$$

so either $\quad (1-\cos 2\pi\alpha) = 0$

or $\quad \underbrace{\frac{c_1}{c_2} = -\frac{c_2}{c_1}}_{\text{not possible}}$

$$\cos 2\pi\alpha = 1$$

$$2\pi\alpha = 2\pi n$$

$$\alpha = n$$

This indicates that we need

$\alpha = \frac{n}{2}$ and $\alpha = n$

this requirement is more restrictive

$\lambda = \alpha^2$

integer

$\lambda = n^2$ ← here, for $n > 0$

but $n = 0$ also works (constant solutions)

so $\boxed{\lambda = n^2 \qquad n = 0, 1, 2, \ldots}$

Another strategy, from the ☆'d equations

$$0 = c_1 \sin 2\pi\alpha + c_2 (1 - \cos 2\pi\alpha)$$

$$0 = c_1 (1 - \cos 2\pi\alpha) - c_2 \sin 2\pi\alpha$$

$$\underbrace{\begin{bmatrix} \sin 2\pi\alpha & 1 - \cos 2\pi\alpha \\ 1 - \cos 2\pi\alpha & -\sin 2\pi\alpha \end{bmatrix}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we require that $\det(\text{---}) = 0$ to get a notrivial solution

The determinant is:

$$-\sin^2 2\pi\alpha - (1 - \cos 2\pi\alpha)^2 = 0$$

$$\underbrace{-\sin^2 2\pi\alpha}_{\leq 0} = \underbrace{(1 - \cos 2\pi\alpha)^2}_{\geq 0}$$

this will only work if both sides are zero

$$2\pi\alpha = 2\pi n$$

$$\alpha = n$$

$$\boxed{\lambda = n^2}$$