10. Homework 10

Exercise 10.1 (4.1.5). *Find the equilibrium values of the system*

$$\begin{cases} \frac{dx}{dt} &= xy^2 - x \\ \frac{dy}{dt} &= x\sin(\pi y). \end{cases}$$

Sol. In order for $\frac{dx}{dt} = 0$, we require $xy^2 - x = 0$. Hence, we must have $x = 0$ or $y^2 - 1 = 0$, which implies $y = \pm 1$. In order for $\frac{dy}{dt} = 0$, we require $x\sin(\pi y) = 0$. Hence, we again require that either $x = 0$ or $\sin(\pi y) = 0$, which just tells us that $y \in \mathbb{Z}$. Combining these pieces of information, we see that both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ vanish at the points $(x, y) = (0, y)$ with y arbitrary, and $(x, y) = (x, 1)$ with x arbitrary and $(x, y) = (x, -1)$ with x arbitrary; these are the equilibrium points of the system.

Exercise 10.2 (4.1.9). *Consider the system of differential equations*

$$\begin{cases} \frac{dx}{dt} &= ax + by \\ \frac{dy}{dt} &= cx + dy. \end{cases}$$

1. *Show that $(x, y) = (0, 0)$ is the only equilibrium point of the above system if $ad - bc \neq 0$.*
2. *Show that the above system has a line of equilibrium points if $ad - bc = 0$.*

Proof. Finding an equilibrium point amounts to solving the system of equations

$$\begin{cases} ax + by &= 0 \\ cx + dy &= 0, \end{cases}$$

which in matrix form can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If $ad - bc = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible and so the above equation only has solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Hence, in this situation, $(x, y) = (0, 0)$ is the only equilbrium point.

Otherwise, $ad - bc = 0$ and the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not invertible. In that case, the rank of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is no more than 1 and so has at least a one dimensional nullspace. This corresponds to the above system of equations having at least a line (and possibly all of $\mathbb{R}^2$) as its solution, and hence at least a line of equilibrium points. □

Exercise 10.3 (4.2.3). *Determine the stability or instability of all solutions of the system of ODEs*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix}\vec{x}.$$

Sol. As implied by Theorem 1 in §4.2, it suffices to consider the stability of the trivial solution $\vec{x} = \vec{0}$, which is determined by the eigenvalues of $\begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix}$. These solve

$$0 = \det\left(\begin{pmatrix} -5 & 3 \\ -1 & 1 \end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix} -5-\lambda & 3 \\ -1 & 1-\lambda \end{pmatrix} = (-5-\lambda)(1-\lambda) + 3 = \lambda^2 + 4\lambda - 2,$$

and so are given by

$$\lambda = \frac{-4 \pm \sqrt{16+8}}{2} = \frac{-4 \pm 2\sqrt{6}}{2} = -2 \pm \sqrt{6}.$$

Since $\sqrt{6} > 2$, one of the values of λ is positive and we conclude that the solutions of the given system are unstable.

Exercise 10.4 (4.2.11)**.** *Determine whether the solutions $x(t) \equiv 0$ and $x(t) \equiv 1$ of the single scalar equation $\frac{dx}{dt} = x(1-x)$ are stable or unstable.*

Sol. We provide a heuristic solution and then give a very explicit argument as well.

Let's first consider the solution $x_1(t) \equiv 0$. If we perturb this solution slightly in the positive direction (say by setting $x(0) = \epsilon > 0$), then notice that $\frac{dx}{dt} > 0$ and the solution moves away from zero. Similarly, if we perturb this solution slightly in the negative direction (say by setting $x(0) = \epsilon < 0$), then notice that $\frac{dx}{dt} < 0$ and the solution again moves further away from zero. Hence, the solution is unstable.

Similarly, let us look at $x_2(t) \equiv 1$. If we perturb this solution slightly in the positive direction (say by setting $x(0) = 1 + \epsilon > 1$), then notice that $\frac{dx}{dt} < 0$ and the solution moves back towards 1. Similarly, if we perturb this solution slightly in the negative direction (say by setting $x(0) = 1 - \epsilon < 1$), then notice that $\frac{dx}{dt} > 0$ and the solution again moves back towards 1. Hence, the solution is stable.

The above argument suffices, but if we want to directly use the definition of stability then notice that we can solve this equation analytically since it is separable. Notice that dividing through by $x(1-x)$ away from $x = 0$ and $x = 1$ we have

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) \\
\frac{1}{x(1-x)}\frac{dx}{dt} &= 1 \\
\left(\frac{1}{x} + \frac{1}{1-x}\right)\frac{dx}{dt} &= 1 \\
\frac{d}{dt}\left(\ln|x| - \ln|1-x|\right) &= 1 \\
\ln\left|\frac{x}{1-x}\right| &= t + c \\
\frac{x}{1-x} &= ke^t \quad k \text{ possibly } < 0 \\
x &= ke^t - xke^t \\
x(1+ke^t) &= ke^t \\
x(t) &= \frac{ke^t}{1+ke^t}.
\end{aligned}$$

Suppose $x(0) = x_0 \notin \{0,1\}$. Then, $x_0 + x_0 k = k$ and $k = \frac{x_0}{1-x_0}$. This tells us what $x(t)$ looks like away from $x = 0$ or $x = 1$. With this solution form in mind, let us consider initial conditions that are not zero or one.

Let's look at $x_1(t) \equiv 0$ first. If x_0 is such that $k > 0$, then $x(t) \to 1$ as $t \to \infty$, and in particular no matter how small we make x_0 (and as such, k), $x(t)$ does not stay close to $x_1(t)$; since we only need one solution that starts close to x_1 and does not return to call the equilibrium solution unstable, we conclude that this solution is unstable.

Next, we consider $x_2(t) \equiv 1$. Let $\epsilon > 0$ be arbitrary and choose $\delta = \frac{2}{\epsilon}$. First, suppose $x_0 > 1 - \delta$. Then, $k > \frac{1-\delta}{\delta} > 0$ and we find

$$|x(t) - x_2(t)| = |x(t) - 1| = \left|\frac{ke^t}{1+ke^t} - 1\right| = \left|\frac{ke^t - 1 - ke^t}{1+ke^t}\right| = \frac{1}{1+ke^t} \le \frac{1}{1+k} < \frac{1}{1+\frac{1-\delta}{\delta}} = \frac{1}{\delta} = \frac{\epsilon}{2} < \epsilon.$$

Next, suppose $x_0 < 1+\delta$. Then, $k < 0$ and $|k| < \frac{1+\delta}{\delta}$. Furthermore, $|k| > 1$ because for some $\alpha > 0$, $k = \frac{1+\alpha}{-\alpha}$ and $|k| = \frac{1+\alpha}{\alpha} > 1$. In particular, the denominator of $x(t)$ never approaches zero, and our solution never blows up. We find then

$$|x(t) - x_2(t)| = |x(t) - 1| = \left|\frac{ke^t}{1+ke^t} - 1\right| = \left|\frac{ke^t - 1 - ke^t}{1+ke^t}\right| = \frac{1}{|k|e^t - 1} \leq \frac{1}{|k|-1} < \frac{1}{\frac{1+\delta}{\delta} - 1} = \frac{1}{\delta} = \frac{\epsilon}{2} < \epsilon.$$

Thus, if $|x_0 - 1| < \delta$, $|x(t) - x_2(t)| < \epsilon$ for all time and the equilibrium solution x_2 is stable.

Exercise 10.5 (4.2.13)**.** *Consider the differential equation $\frac{dx}{dt} = x^2$. Show that all solutions $x(t)$ with $x(0) \geq 0$ are unstable while all solutions $x(t)$ with $x(0) < 0$ are asymptotically stable.*

Proof. First consider a solution $x(0) \geq 0$. A heuristic argument for instability is that for small positive perturbations away from zero (i.e. $y(0) = x(0) + \epsilon > 0$), $\frac{dx}{dt} > 0$ and so solutions grow further away from zero. We can prove this directly by looking at analytic solutions for the equation away from zero. Notice that for x away from zero, we can write

$$\begin{aligned} \frac{1}{x^2}\frac{dx}{dt} &= 1 \\ \frac{d}{dt}\left(-\frac{1}{x}\right) &= 1 \\ -\frac{1}{x} &= t + c \\ x(t) &= \frac{-1}{t+c}. \end{aligned}$$

If $x(0) = 0$ then $x(t) \equiv 0$ for all $t > 0$ and if $x(0) = x_0 \neq 0$, we have

$$x_0 = \frac{-1}{c} \implies x(t) = \frac{-1}{t - \frac{1}{x_0}} = \frac{1}{\frac{1}{x_0} - t}.$$

Suppose that $x_0 \geq 0$. Let $y(0) = x_0 + \delta > 0$ for $\delta > 0$ arbitrarily small. Then, as $t \to \frac{1}{x_0+\delta}$, $y(t) \uparrow \infty$. However, $x\left(\frac{1}{x_0+\delta}\right) < \infty$ since $x(t)$ does not blow up until $t = \frac{1}{x_0}$ (or never, in the case $x_0 = 0$). In particular, as $t \to \frac{1}{x_0+\delta}$, $|x(y) - y(t)| \to \infty$ and so solutions will not stay close to $x(t)$ no matter how small the perturbation is. It follows that solutions $x(t)$ with $x(0) \geq 0$ are unstable.

Let us now consider solution $x(t)$ with $x(0) = x_0 < 0$. Then, by the above we can write

$$x(t) = \frac{1}{\frac{1}{x_0} - t} = \frac{1}{t + \frac{1}{|x_0|}}.$$

Suppose $|y(0) - x_0| < \frac{|x_0|}{2}$, so that in particular $y(0) < 0$ and $|y(0)| > \frac{|x_0|}{2}$. Then

$$y(t) = \frac{1}{t + \frac{1}{|y(0)|}}$$

and

$$\begin{aligned} |x(t) - y(t)| &= \left|\frac{1}{t + \frac{1}{|y(0)|}} - \frac{1}{t + \frac{1}{|x_0|}}\right| \\ &= \left|\frac{t + \frac{1}{|x_0|} - t - \frac{1}{|y(0)|}}{\left(t + \frac{1}{|y(0)|}\right)\left(t + \frac{1}{|x_0|}\right)}\right| \\ &\leq \frac{|y(0) - x_0|}{|y(0)||x_0|}\frac{1}{t^2} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{|x_0|}{2\frac{|x_0|}{2}|x_0|}\frac{1}{t^2} \\
&= \frac{1}{|x_0|}\frac{1}{t^2} \to 0.
\end{aligned}$$

so that $|x(t) - y(t)| \to 0$ as $t \to \infty$ and $y(t)$ ultimately returns to $x(t)$. In particular, the solution $x(t)$ is asymptotically stable. □

Exercise 10.6 (4.3.3)**.** *Find all equilibrium solutions of*

$$\begin{cases} \frac{dx}{dt} &= x^2 + y^2 - 1 \\ \frac{dy}{dt} &= 2xy \end{cases}$$

and determine, if possible, whether they are stable or unstable.

Sol. First, observe that we can write the system as $\frac{d}{dt}\vec{x} = \mathbf{F}(\vec{x})$, with $\vec{x} = (x, y)$ and

$$\mathbf{F}(\vec{x}) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - 1 \\ 2xy \end{pmatrix}.$$

Notice that $\frac{dy}{dt} = 0$ if and only if at least one of x or y is zero, and that $\frac{dx}{dt} = 0$ if and only if $x^2 + y^2 = 1$. Thus, the four points $(0, 1)$, $(0, -1)$, $(1, 0)$ and $(-1, 0)$ are the four equilibrium points of this system.

Let us first expand the system about $(0, 1)$, and let $\vec{x} = (x, y)$, $\vec{z} = \vec{x} - (0, 1)$. Using a Taylor series expansion, we have that

$$\begin{aligned}
\frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Big|_{(0,1)}\vec{z} + \mathbf{g}(\vec{z}) \\
&= \begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix}\Big|_{(0,1)}\vec{z} + \mathbf{g}(\vec{z}) \\
&= \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\vec{z} + \mathbf{g}(\vec{z})
\end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix} -\lambda & 2 \\ 2 & -\lambda \end{pmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2),$$

or $\lambda = \pm 2$. Since one of the eigenvalues of A is positive, we conclude by Theorem 2 in §4.3 that this equilibrium solution is unstable.

Similarly, we consider $(0, -1)$. We let $\vec{x} = (x, y)$, $\vec{z} = \vec{x} - (0, -1)$. Using a Taylor series expansion, we have that

$$\begin{aligned}
\frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}\Big|_{(0,-1)}\vec{z} + \mathbf{g}(\vec{z}) \\
&= \begin{pmatrix} 2x & 2y \\ 2y & 2x \end{pmatrix}\Big|_{(0,-1)}\vec{z} + \mathbf{g}(\vec{z}) \\
&= \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}).
\end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix} -\lambda & -2 \\ -2 & -\lambda \end{pmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2),$$

or $\lambda = \pm 2$. Since one of the eigenvalues of A is again positive, we conclude by Theorem 2 in §4.3 that this equilibrium solution is unstable.

Next, we consider $(1,0)$. We let $\vec{x} = (x,y)$, $\vec{z} = \vec{x} - (1,0)$. Using a Taylor series expansion, we have that

$$\begin{aligned} \frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\Big|_{(1,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix}2x & 2y\\ 2y & 2x\end{pmatrix}\Big|_{(1,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}). \end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix}2-\lambda & 0\\ 0 & 2-\lambda\end{pmatrix} = (2-\lambda)^2,$$

or $\lambda = 2$. Since one of the eigenvalues of A is again positive, we conclude by Theorem 2 in §4.3 that this equilibrium solution is unstable.

Finally, we consider $(-1,0)$. We let $\vec{x} = (x,y)$, $\vec{z} = \vec{x} - (-1,0)$. Using a Taylor series expansion, we have that

$$\begin{aligned} \frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\Big|_{(-1,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix}2x & 2y\\ 2y & 2x\end{pmatrix}\Big|_{(-1,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}). \end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix}-2-\lambda & 0\\ 0 & -2-\lambda\end{pmatrix} = (2+\lambda)^2,$$

or $\lambda = -2$. Since both of the eigenvalues of A are negative, we conclude by Theorem 2 in §4.3 that this equilibrium solution is in fact stable.

Exercise 10.7 (4.3.9)**.** *Verify that the origin is an equilibrium point of*

$$\begin{cases} \frac{dx}{dt} &= e^{x+y} - 1 \\ \frac{dy}{dt} &= \sin(x+y) \end{cases}$$

and determine, if possible, if it is stable or unstable.

Sol. We first observe that at $(0,0)$, we have $\frac{dx}{dt} = e^0 - 1 = 1 - 1 = 0$ and $\frac{dy}{dt} = \sin(0+0) = 0$ so that the origin is indeed an equilibrium point. To determine whether or not it is stable or unstable, we notice that this equation is of the form $\frac{d}{dt}\vec{x} = \mathbf{F}(\vec{x})$ with $\vec{x} = (x,y)$ and

$$\mathbf{F}(\vec{x}) = \begin{pmatrix}f_1(x,y)\\ f_2(x,y)\end{pmatrix} = \begin{pmatrix}e^{x+y} - 1\\ \sin(x+y)\end{pmatrix}.$$

Taylor expanding the solution about $(0,0)$, we have with $\vec{z} = \vec{x} - (0,0)$ that

$$\begin{aligned} \frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix}e^{x+y} & e^{x+y}\\ \cos(x+y) & \cos(x+y)\end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}). \end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix}1-\lambda & 1\\ 1 & 1-\lambda\end{pmatrix} = (1-\lambda)^2 - 1 = -2\lambda + \lambda^2 = \lambda(\lambda-2),$$

i.e. $\lambda = 0$ or $\lambda = 2$. Since one of the eigenvalues is positive, we conclude by Theorem 2 in §4.3 that this equilibrium is unstable.

Exercise 10.8 (4.3.11)**.** *Verify that the origin is an equilibrium point of*

$$\begin{cases} \frac{dx}{dt} &= \cos y - \sin x - 1 \\ \frac{dy}{dt} &= x - y - y^2 \end{cases}$$

and determine, if possible, if it is stable or unstable.

Sol. We first observe that at $(0,0)$, we have $\frac{dx}{dt} = \cos(0) - \sin(0) - 1 = 1 - 0 - 1 = 0$ and $\frac{dy}{dt} = 0 - 0 - 0^2 = 0$ so that the origin is indeed an equilibrium point. To determine whether or not it is stable or unstable, we notice that this equation is of the form $\frac{d}{dt}\vec{x} = \mathbf{F}(\vec{x})$ with $\vec{x} = (x,y)$ and

$$\mathbf{F}(\vec{x}) = \begin{pmatrix}f_1(x,y)\\ f_2(x,y)\end{pmatrix} = \begin{pmatrix}\cos y - \sin x - 1\\ x - y - y^2\end{pmatrix}.$$

Taylor expanding the solution about $(0,0)$, we have with $\vec{z} = \vec{x} - (0,0)$ that

$$\begin{aligned} \frac{d}{dt}\vec{z} = \frac{d}{dt}\vec{x} &= \mathbf{F}\begin{pmatrix}0\\0\end{pmatrix} + \begin{pmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix}-\cos x & -\sin y\\ 1 & -1-2y\end{pmatrix}\Big|_{(0,0)}\vec{z} + \mathbf{g}(\vec{z}) \\ &= \begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}\vec{z} + \mathbf{g}(\vec{z}). \end{aligned}$$

with $\frac{|\mathbf{g}(z)|}{|z|} \to 0$ as $|z| \to 0$. Notice that the eigenvalues of $\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix}$ are solutions of

$$0 = \det\left(\begin{pmatrix}-1 & 0\\ 1 & -1\end{pmatrix} - \lambda\mathbf{I}\right) = \det\begin{pmatrix}-1-\lambda & 0\\ 1 & -1-\lambda\end{pmatrix} = (1+\lambda)^2,$$

with solution $\lambda = -1$. Since all eigenvalues are all negative, we conclude by Theorem 2 in §4.3 that the equilibrium point is stable.

11. Homework 11

Exercise 11.1 (4.4.1). *Verify that*

$$\begin{cases} x(t) = 1 + t \\ y(t) = \cos(t^2) \end{cases}$$

is a solution of

$$\begin{cases} \dot{x} = 1 \\ \dot{y} = 2(1 - x)\sin(1 - x)^2 \end{cases}$$

and find its orbit.

Sol. We first observe that

$$\frac{dx}{dt} = 1$$

and

$$\frac{dy}{dt} = -2t\sin(t^2) = -2(x-1)\sin(x-1)^{)} = 2(1-x)\sin(1-x)^2$$

with $t = x - 1$. It's orbit is simply the trajectory traced out by the corresponding curve $\Phi(x, y) = 0$ parametrized by $x(t)$ and $y(t)$, which can be found by substituting $t = x - 1$:

$$y(x) = \cos(x-1)^2.$$

Exercise 11.2 (4.4.5). *Find the orbits of*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x. \end{cases}$$

Sol. Notice first that the only equilibrium point $\dot{x} = \dot{y} = 0$ is given by $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Away from this equilibrium, trajectories satisfy the first order differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{-x}{y}.$$

This equation is separable, and we can solve

$$\begin{aligned} y\frac{dy}{dx} &= -x \\ \frac{d}{dx}\left(\frac{1}{2}y^2\right) &= -x \\ \frac{1}{2}y^2 = -\frac{1}{2}x^2 &+ c \\ x^2 + y^2 &= c. \end{aligned}$$

So, we see that the trajectories are the equilibrium point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the circles $x^2 + y^2 = c$.

Exercise 11.3 (4.4.13). *Find the orbits of*

$$\begin{cases} \dot{x} = 2xy \\ \dot{y} = x^2 - y^2. \end{cases}$$

Sol. We first find the equilibrium points, which satisfy $\dot{x} = \dot{y} = 0$. So, $2xy = 0$ which implies that x or y is zero, which coupled with $x^2 - y^2 = 0$ forces both x and y to be zero. Outside of the equilibrium point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, trajectories satisfy the first order differential equation

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x^2 - y^2}{2xy}.$$

We can rewrite this equation as

$$(y^2 - x^2) + 2xy\frac{dy}{dx} = 0,$$

which is exact since $\frac{\partial}{\partial y}\left(y^2 - x^2\right) = 2y = \frac{\partial}{\partial x}(2xy)$. Integrating $2xy$ with respect to y, we find that the solution $\Phi(x, y) = c$ has to satisfy

$$\Phi(x, y) = \int 2xy \; dy = xy^2 + k(x)$$

for some $k(x)$. Differentiating with respect to x,

$$y^2 - x^2 = \frac{\partial \Phi}{\partial x} = y^2 + k'(x),$$

so $k'(x) = -x^2$ and $k(x) = -\frac{1}{3}x^3 + c$. Thus, the trajectories are given by the equilibrium point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and the curves $xy^2 - \frac{1}{3}x^3 = c$.

Exercise 11.4 (4.7.1)**.** *Draw the phase portrait of*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix} \vec{x}.$$

Sol. We use the eigenmethod, and find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} -5 & 1 \\ 1 & -5 \end{pmatrix}$. The eigenvalues λ must satisfy

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -5-\lambda & 1 \\ 1 & -5-\lambda \end{vmatrix} = (5+\lambda)^2 - 1 = \lambda^2 + 10\lambda + 24 = (\lambda+4)(\lambda+6),$$

and so the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -6$.

Now, for $\lambda_1 = -4$ our eigenvector $\vec{v}_1$ must satisfy $(\mathbf{A} + 4\mathbf{I})\vec{v}_1 = \vec{0}$. Row reducing, we have

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so any eigenvector is a multiple of $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -6$, our eigenvector $\vec{v}_2$ must satisfy $(\mathbf{A} + 6\mathbf{I})\vec{v}_2 = \vec{0}$. Row reducing, we have

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so any eigenvector is a multiple of $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

It follows that the general solution is $\vec{x}(t) = c_1 e^{-4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Solutions with $c_1 = 0$ converge asymptotically to the equilibrium point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, but otherwise solutions converge asymptotically to equilibrium approaching the vector $\begin{pmatrix}1\\1\end{pmatrix}$ since $-4 > -6$. The phase diagram is given in Figure 1 in the handwritten notes at the end of this document.

Exercise 11.5 (4.7.5)**.** *Draw the phase portrait of*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix}1 & -4\\-8 & 4\end{pmatrix}\vec{x}.$$

Sol. We use the eigenmethod, and find the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix}1 & -4\\-8 & 4\end{pmatrix}$. The eigenvalues λ must satisfy

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix}1-\lambda & -4\\-8 & 4-\lambda\end{vmatrix} = (1-\lambda)(4-\lambda) - 32 = \lambda^2 - 5\lambda - 28,$$

and so the eigenvalues are $\lambda_1 = \frac{5+\sqrt{137}}{2}$ and $\lambda_2 = \frac{5-\sqrt{137}}{2}$.

Now, for $\lambda_1 = \frac{5+\sqrt{137}}{2}$ our eigenvector $\vec{v}_1$ must satisfy $(\mathbf{A} - \lambda_1\mathbf{I})\vec{v}_1 = \vec{0}$. Row reducing, we have

$$\begin{pmatrix}\frac{-3-\sqrt{137}}{2} & -4 & 0\\-8 & \frac{3-\sqrt{137}}{2} & 0\end{pmatrix} \to \begin{pmatrix}\frac{3+\sqrt{137}}{8} & 1 & 0\\0 & 0 & 0\end{pmatrix},$$

so any eigenvector is a multiple of $\vec{v}_1 = \begin{pmatrix}1\\\frac{-3-\sqrt{137}}{8}\end{pmatrix}$.

For $\lambda_2 = \frac{5-\sqrt{137}}{2}$, our eigenvector $\vec{v}_2$ must satisfy $(\mathbf{A} - \lambda_2\mathbf{I})\vec{v}_2 = \vec{0}$. Row reducing, we have

$$\begin{pmatrix}\frac{-3+\sqrt{137}}{2} & -4 & 0\\-8 & \frac{3+\sqrt{137}}{2} & 0\end{pmatrix} \to \begin{pmatrix}\frac{3-\sqrt{137}}{8} & 1 & 0\\0 & 0 & 0\end{pmatrix},$$

so any eigenvector is a multiple of $\vec{v}_2 = \begin{pmatrix}1\\\frac{-3+\sqrt{137}}{8}\end{pmatrix}$.

It follows that the general solution is $\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2$. Observe that $\lambda_1 > 0 > \lambda_2$. Solutions with $c_1 = 0$ converge asymptotically to the equilibrium point $\begin{pmatrix}0\\0\end{pmatrix}$ along the vector $\vec{v}_2$, but otherwise solutions diverge from equilibrium approaching the vector $\vec{v}_1$ since $\lambda_1 > 0$. The phase diagram is given in Figure 2 in the handwritten notes at the end of this document.

Exercise 11.6 (4.7.11)**.** *The equation of motion of a spring-mass system with damping (see Section 2.6) is $m\ddot{z} + c\dot{z} + kz = 0$, where m, c, and k are positive numbers. Convert this equation to a system of first-order equations for $x = z$, $y = \dot{z}$, and draw the phase portrait of this system. Distinguish the overdamped, critically damped, and underdamped cases.*

Sol. Observe that with $x = z$ and $y = \dot{z}$, we have the system of equations

$$\frac{d\vec{x}}{dt} = \begin{pmatrix}\dot{x}\\\dot{y}\end{pmatrix} = \begin{pmatrix}\dot{z}\\\ddot{z}\end{pmatrix} = \begin{pmatrix}y\\-\frac{k}{m}x - \frac{c}{m}y\end{pmatrix} = \begin{pmatrix}0 & 1\\-\frac{k}{m} & -\frac{c}{m}\end{pmatrix}\vec{x}.$$

To determine the phase portrait of this system, we need to characterize the eigenvalues which govern the solutions. These solve the characteristic equation

$$0 = \det\left(\begin{pmatrix}0 & 1\\-\frac{k}{m} & -\frac{c}{m}\end{pmatrix} - \lambda\mathbf{I}\right) = \begin{vmatrix}-\lambda & 1\\-\frac{k}{m} & -\frac{c}{m} - \lambda\end{vmatrix} = \lambda\left(\lambda + \frac{c}{m}\right) + \frac{k}{m} = \lambda^2 + \frac{c}{m}\lambda + \frac{k}{m},$$

whose solutions $m\lambda^2 + c\lambda + k = 0$ satisfy $\lambda = \frac{-c\pm\sqrt{c^2-4km}}{2m}$. We distinguish three cases.

Case 1: $c^2 - 4km > 0$, or **overdamped** motion.

In this case, observe that both eigenvalues λ are real. Furthermore, $c^2 - 4km < c^2$ since $k, m, c > 0$, so $\sqrt{c^2 - 4km} < c$ and both eigenvalues are negative. Hence, we have asymptotic convergence to equilibrium $\begin{pmatrix}0\\0\end{pmatrix}$ along the corresponding eigenvectors, as in Figure 3a of the handwritten notes at the end of this document.

Case 2: $c^2 - 4km = 0$, or **critically damped** motion.

Notice that we have a double root $\lambda = \frac{-c}{2m}$. We are only able to find a single linearly independent eigenvector though, since row reduction yields solutions to $\left(\begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{pmatrix} - \lambda\mathbf{I}\right)\vec{v} = \vec{0}$ satisfying

$$\begin{pmatrix}\frac{c}{2m} & 1 & 0\\ -\frac{k}{m} & -\frac{c}{2m} & 0\end{pmatrix} \to \begin{pmatrix}1 & \frac{2m}{c} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

The components a and b of $\vec{v}$ must solve $a + \frac{2m}{c}b = 0$, which means that we only have one free parameter to determine the eigenvalue and hence a one dimensional eigenspace. Let's call the eigenvector $\vec{v}$; then, we must look for a linearly independent generalized eigenvector $\vec{u}$ satisfying $\left(\begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{pmatrix} - \lambda\mathbf{I}\right)^2\vec{u} = \vec{0}$, and our general solution is of the form $\vec{x}(t) = c_1e^{\lambda t}\vec{v} + c_2e^{\lambda t}\left(\vec{u} + \left(\begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{pmatrix} - \lambda\mathbf{I}\right)t\vec{u}\right)$. Since we are only in two dimensions and $\left(\begin{pmatrix}0 & 1\\ -\frac{k}{m} & -\frac{c}{m}\end{pmatrix} - \lambda\mathbf{I}\right)\vec{u}$ is linearly independent from $\vec{u}$, it must be some constant k times $\vec{v}$, i.e. $k\vec{v}$. Thus, we can write

$$\vec{x}(t) = c_1e^{\lambda t}\vec{v} + c_2e^{\lambda t}(\vec{u} + kt\vec{v}) = (c_1 + c_2kt)e^{\lambda t}\vec{v} + c_2e^{\lambda t}\vec{u}.$$

Notice that since c_2kt asymptotically dominates c_2, even in the $c_1 = 0$ case we find asymptotic convergence to the equilibrium point $\begin{pmatrix}0\\0\end{pmatrix}$ along $\vec{v}$ as in Figure 3b of the handwritten notes at the end of this document.

Case 3: $c^2 - 4km < 0$, or **underdamped motion**.

Notice that we have two complex eigenvalues with negative imaginary part, so trajectories spiral towards the equilibrium point $\begin{pmatrix}0\\0\end{pmatrix}$. Furthermore, notice that at $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\y\end{pmatrix}$ for positive y, $\frac{dx}{dt} = y > 0$, and so the spirals move clockwise as in Figure 3c of the handwritten notes at the end of this document.

Exercise 11.7 (4.7.13)**.** *In this problem, we consider the system*

$$\begin{cases}\dot{x} = y\\ \dot{y} = x + 2x^3.\end{cases}$$

1. *Show that the equilibrium solution $x = 0$, $y = 0$ of the linearized system is a saddle, and draw the phase portrait of the linearized system.*

2. *Find the orbits of the given system, and draw its phase portrait.*

3. *Show that there are exactly two orbits of the the nonlinear system (one for $x > 0$ and one for $x < 0$) on which $x \to 0$, $y \to 0$ as $t \to \infty$. Similarly, there are exactly two orbits of the nonlinear system on which $x \to 0$, $y \to 0$ as $t \to -\infty$. Thus, observe that the phase portraits for the nonlinear and linearized systems look the same near the origin.*

Sol. We first determine the linearized system about $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, which is given by

$$\frac{d\vec{z}}{dt} = \begin{pmatrix} \frac{\partial}{\partial x}(y) & \frac{\partial}{\partial y}(y) \\ \frac{\partial}{\partial x}(x+2x^3) & \frac{\partial}{\partial y}(x+2x^3) \end{pmatrix} \Big|_{(0,0)} \vec{z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \vec{z}.$$

To determine the phase portrait of this system, we consider the eigenvalues and eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The eigenvalues solve

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1,$$

or $\lambda = \pm 1$. Since $\lambda_1 = 1 > 0 > -1 = \lambda_2$, the equilibrium solution is a saddle. To draw the phase portrait, we determine the associated eigenvectors. For $\lambda_1 = 1$, we need $(\mathbf{A} - \mathbf{I})\vec{v}_1 = \vec{0}$ and so by row reduction we see that

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that any eigenvector is a multiple of $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda_2 = -1$, we need $(\mathbf{A} + \mathbf{I})\vec{v}_2 = \vec{0}$ and so by row reduction we see that

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that any eigenvector is a multiple of $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

It follows that the general solution is given by $\vec{z}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Solutions with $c_1 = 0$ converge asymptotically to the equilibrium solution $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ along the vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and otherwise diverge from equilibrium approaching the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The associated phase portrait is given in Figure 4a of the handwritten notes at the end of this document.

For the nonlinear system, away from the equilibrium point $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ trajectories satisfy the first order ODE

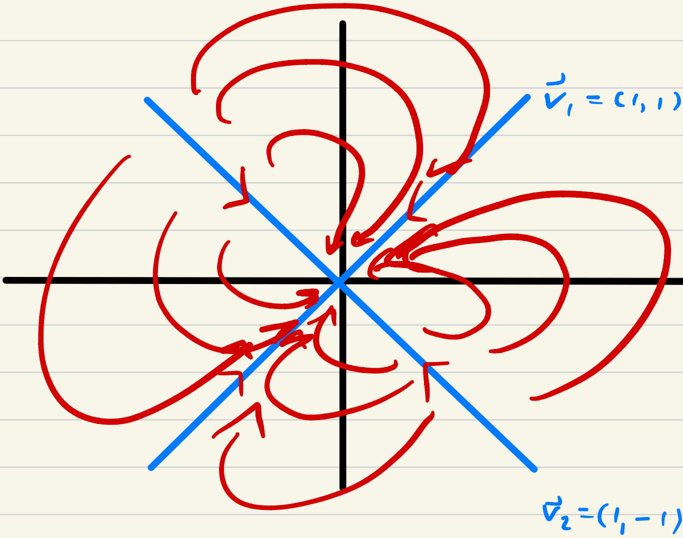
$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x + 2x^3}{y}.$$

This equation is separable, and has solutions

$$\begin{aligned} y\frac{dy}{dx} &= x + 2x^3 \\ \frac{d}{dx}\left(\frac{1}{2}y^2\right) &= x + 2x^3 \\ \frac{1}{2}y^2 &= \frac{1}{2}x^2 + \frac{1}{2}x^4 + c, \end{aligned}$$

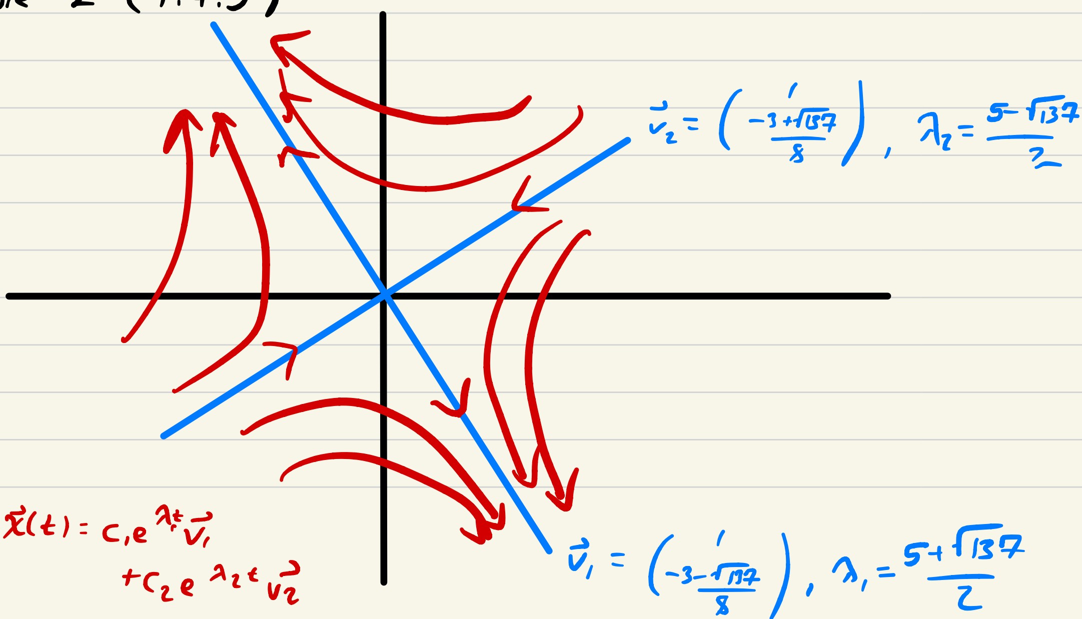
so trajectories satisfy $y^2 = x^2 + x^4 + c$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The phase portrait is given in Figure 4b of the handwritten notes at the end of this document, as is the identification of the requested orbits that converge to equilibrium as $t \to \infty$ and $t \to -\infty$.

- Figure 1 (4.7.1)



$$\vec{x}(t) = c_1 e^{-4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-6t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

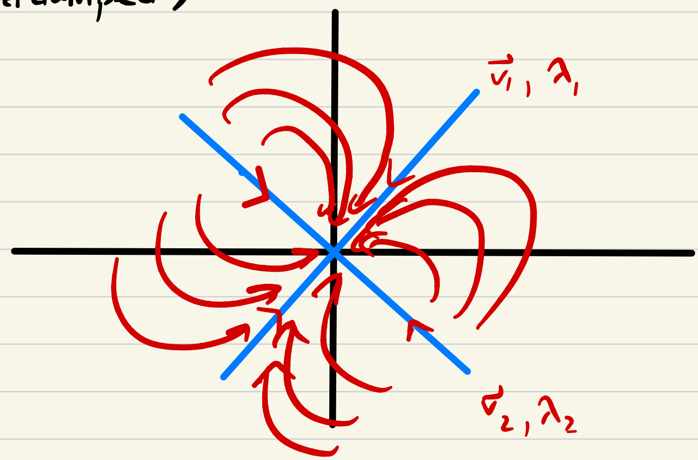
- Figure 2 (4.7.5)



$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$

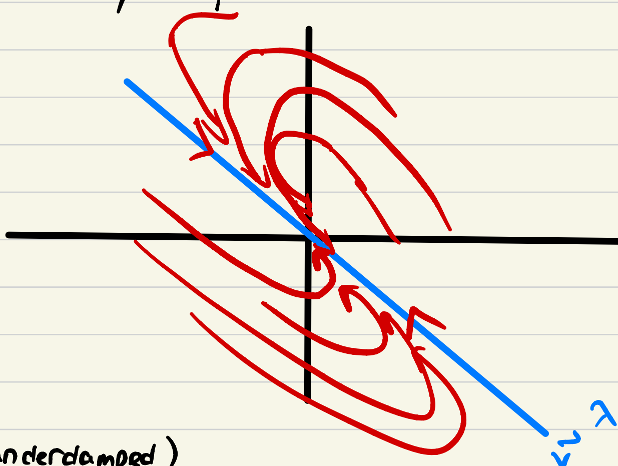
- Figure 3 (4.7.11)

- 3a (overdamped)



$0 > \lambda_1 > \lambda_2$

- 3b (critically damped)



$\lambda < 0$

double root

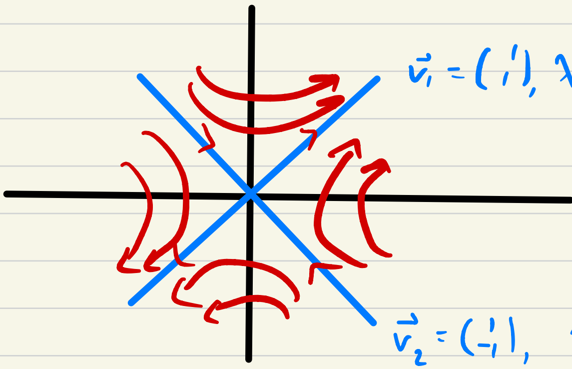
- 3c (underdamped)

$\lambda_1, \lambda_2 \in \mathbb{C}$

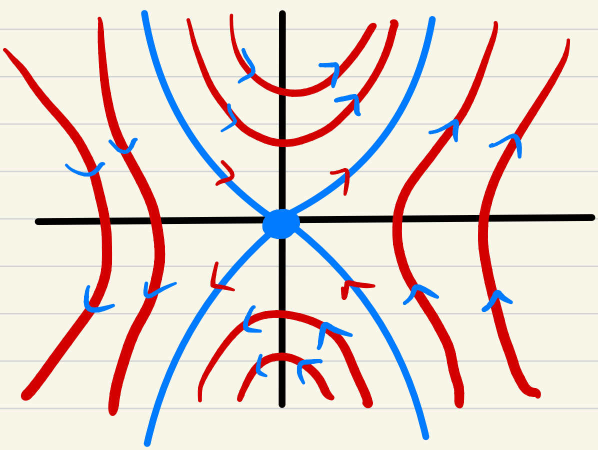
$Re(\lambda_1)$,

$Re(\lambda_2) < 0$

- Figure 4 (4.7.13)
  - 4a (linearized orbits)



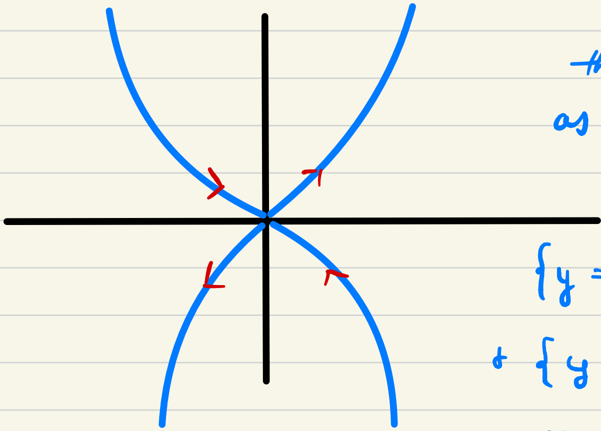
  - 4b (nonlinear orbits)



$y^2 = x^2 + x^4 + c$

for $x > 0$, $\dot{y} > 0$;
for $x < 0$, $\dot{y} < 0$.
Hence, the direction of orbit.

Notice that for $x, y \to 0$, we need $c = 0$ (otherwise, $\|\binom{x}{y}\|$ is bounded away from zero). Hence, we have the four branches of $x^2 + x^4 = y^2$, $(x, y) \neq (0, 0)$:



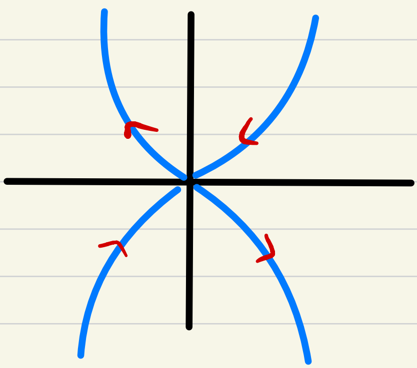
The arrows from the above diagram are as we move forward in time, $t \to \infty$. Thus,

$$\{y = \sqrt{x^2 + x^4},\ x < 0\}$$

$$+ \{y = -\sqrt{x^2 + x^4},\ x > 0\}$$

converge to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to \infty$.

As $t \to -\infty$, we reverse the arrows ($s = -t$, $\frac{dy}{ds} = \frac{dy}{dt}\frac{dt}{ds} = -\frac{dy}{dt}$; similarly for $\frac{dx}{ds}$). Thus, as $t \to -\infty$, the orbits near $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ look like:



So,

$$\{y = \sqrt{x^2 + x^4},\ x > 0\}$$

$$+ \{y = -\sqrt{x^2 + x^4},\ x < 0\}$$

converge to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as $t \to -\infty$.

Furthermore, for small $\begin{pmatrix} x \\ y \end{pmatrix}$, $x^4 \ll x^2$ and so $y^2 \approx x^2$, or $y \approx \pm x$. In particular, the orbits for the nonlinear problem look like the lines we saw in the linearized orbits near our equilibrium point!!