8. Homework 8

Exercise 8.1 (3.1.1). *Convert the differential equation*

$$\frac{d^3y}{dt^3} + \left(\frac{dy}{dt}\right)^2 = 0$$

into a system of first-order equations.

Sol. Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$. Then, we observe from the above equation that

$$\begin{aligned}\frac{dx_1}{dt} &= y' = x_2\\ \frac{dx_2}{dt} &= y'' = x_3\\ \frac{dx_3}{dt} &= y''' = -(y')^2 = -x_2^2,\end{aligned}$$

which is a system of first-order equations.

Exercise 8.2 (3.1.5). *1. Let $y(t)$ be a solution of the equation $y'' + y' + y = 0$. Show that*

$$\vec{x}(t) = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

is a solution of the system of equations

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

2. Let

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

be a solution of the system of equations

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

Show that $y = x_1(t)$ is a solution of the equation $y'' + y' + y = 0$.

Proof. Let us consider the first point first. Notice that with $x_1 = y$, $x_2 = y'$,

$$\begin{aligned}\frac{dx_1}{dt} &= y' = x_2\\ \frac{dx_2}{dt} &= y'' = -y - y' = -x_1 - x_2.\end{aligned}$$

In matrix form, this system is exactly

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x},$$

as desired.

For the second point, suppose

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

is a solution of

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

and set $y = x_1$. Then, we observe that $x_2 = \frac{dx_1}{dt} = y'$ and

$$\frac{dx_2}{dt} = y'' = -x_1 - x_2 = -y - y',$$

so that $y'' + y' + y = 0$. Hence, $y = x_1(t)$ solves the desired equation. □

Exercise 8.3 (3.1.7)**.** *Write the system of differential equations and initial values*

$$\begin{cases} \frac{dx_1}{dt} = 5x_1 + 5x_2, & x_1(3) = 0 \\ \frac{dx_2}{dt} = -x_1 + 7x_2, & x_2(3) = 6 \end{cases}$$

in the form $\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$, $\vec{x}(t_0) = \vec{x}_0$.

Sol. We immediately observe that we have

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix} \vec{x}$$

and that our initial condition takes the form

$$\vec{x}(3) = \begin{pmatrix} x_1(3) \\ x_2(3) \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}.$$

Exercise 8.4 (3.8.1)**.** *Find all solutions of the system*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \vec{x}.$$

Sol. We use the eigenmethod, searching for solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ for eigenvalues λ of the matrix $\mathbf{A}$ and associated eigenvector $\vec{v}$. Observe that the characteristic equation is of the form

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 6-\lambda & -3 \\ 2 & 1-\lambda \end{vmatrix} = (6-\lambda)(1-\lambda) + 6 = \lambda^2 - 7\lambda + 12 = (\lambda - 4)(\lambda - 3),$$

with eigenvalue solutions $\lambda = 4$ and $\lambda = 3$. It remains to determine the associated linearly independent eigenvectors.

First, we examine $\lambda = 4$. This requires us to solve

$$\begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6v_1 - 3v_2 \\ 2v_1 + v_2 \end{pmatrix} = \begin{pmatrix} 4v_1 \\ 4v_2 \end{pmatrix}.$$

The second equation can be written $2v_1 = 3v_2$, or $v_2 = (2/3)v_1$. Sure enough, the first equation takes the same form. Hence, v_1 is free to choose; we pick $v_1 = 3$, so $v_2 = 2$ and our eigenvector is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Next, we examine $\lambda = 3$. This requires us to solve

$$\begin{pmatrix} 6 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 6v_1 - 3v_2 \\ 2v_1 + v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 \\ 3v_2 \end{pmatrix}.$$

The second equation gives $2v_1 = 2v_2$, or $v_1 = v_2$. Sure enough, the first equation takes the same form. Hence, v_1 is free to choose; we pick $v_1 = 1$, so $v_2 = 1$ and our eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus, our general solution is

$$\vec{x} = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Exercise 8.5 (3.8.11)**.** *Solve the initial-value problem*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{pmatrix} \vec{x}, \quad \vec{x}(0) = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}.$$

Sol. We use the eigenmethod, searching for solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ for eigenvalues λ of the matrix $\mathbf{A}$ and associated eigenvector $\vec{v}$. Observe that the characteristic equation is of the form

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & -3 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & -1 & -2-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(-2-\lambda),$$

with solutions $\lambda = 1$, $\lambda = -1$, and $\lambda = -2$. We could find all of the solutions, but instead we notice a special similarity between the eigenvector for $\lambda = -2$ and the initial conditions. Namely, if we start trying to find an eigenvector for $\lambda = -2$, we observe

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v} = \begin{pmatrix} 3 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 3v_1 - 3v_2 + 2v_3 \\ v_2 \\ -v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

In particular, $v_2 = 0$ and $3v_1 = -2v_3$, or $v_1 = (-2/3)v_3$. Notice then that v_3 is free to choose, so we might as well set $v_3 = 3$ and find that $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \vec{x}(0)$ is an eigenvector for $\lambda = -2$! Hence, if we were to write out a general solution

$$\vec{x}(t) = c_1 e^t \vec{v}_1 + c_2 e^{-t} \vec{v}_2 + c_3 e^{-2t} \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}$$

with eigenvector $\vec{v}_1$ for $\lambda = 1$ and $\vec{v}_2$ for $\lambda = -1$, we would find by setting $t = 0$ that

$$\vec{x}(0) = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3 \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix},$$

so $c_1 = c_2 = 0$ and $c_3 = 1$. Hence, we needn't bother finding other eigenvectors and can see immediately that the solution to this initial value problem is

$$\vec{x}(t) = e^{-2t} \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix}.$$

Exercise 8.6 (3.9.1)**.** *Find the general solution of the system*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \vec{x}.$$

Sol. We use the eigenmethod, searching for solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ for eigenvalues λ of the matrix $\mathbf{A}$ and associated eigenvector $\vec{v}$. Observe that the characteristic equation is of the form

$$0 = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{vmatrix} = (3+\lambda)(1+\lambda) + 2 = \lambda^2 + 4\lambda + 5,$$

with solutions $\lambda = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i$. These are complex eigenvalues, so we will choose one and find the real and imaginary parts of $e^{\lambda t}\vec{v}$. Choosing $\lambda = -2 + i$, we seek a vector $\vec{v}$ satisfying

$$\begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -3v_1 + 2v_2 \\ -v_1 - v_2 \end{pmatrix} = \begin{pmatrix} (-2+i)v_1 \\ (-2+i)v_2 \end{pmatrix}.$$

The second equation yields $v_1 = (1-i)v_2$. Similarly, the first equation gives $-3v_1 + 2v_2 = -2v_1 + iv_1$ or $(1+i)v_1 = 2v_2$. In particular, $v_1 = \frac{2}{1+i}v_2 = \frac{2(1-i)}{1+1}v_2 = (1-i)v_2$, which is the same equation we had. Hence, v_2 is free to choose. Choosing $v_2 = 1$, our eigenvector is $\vec{v} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}$, and we see that

$$\begin{aligned} e^{\lambda t}\vec{v} &= e^{(-2+i)t}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \\ &= e^{-2t}(\cos(t) + i\sin(t))\begin{pmatrix} 1-i \\ 1 \end{pmatrix} \\ &= e^{-2t}\begin{pmatrix} \cos(t) - i\cos(t) + i\sin(t) + \sin(t) \\ \cos(t) + i\sin(t) \end{pmatrix}. \end{aligned}$$

Taking real and imaginary parts, we find that our general solution is of the form

$$\vec{x}(t) = e^{-2t}\left(c_1\begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{pmatrix} + c_2\begin{pmatrix} \sin(t) - \cos(t) \\ \sin(t) \end{pmatrix}\right).$$

Exercise 8.7 (3.9.7)**.** *Solve the initial value problem*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}.$$

Sol. This is...messy. But at least the ideas are straightforward. We again use the eigenmethod, searching for solutions $\vec{x}(t) = e^{\lambda t}\vec{v}$ for eigenvalues λ of the matrix $\mathbf{A}$ and associated eigenvector $\vec{v}$. Observe that the characteristic equation is of the form

$$\begin{aligned} 0 = \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -3-\lambda & 0 & 2 \\ 1 & -1-\lambda & 0 \\ -2 & -1 & -\lambda \end{vmatrix} \\ &= (-3-\lambda)(-\lambda(-1-\lambda)) + 2(-1 + 2(-1-\lambda)) \\ &= -\lambda(\lambda^2 + 4\lambda + 3) - 6 - 4\lambda \\ &= -\lambda^3 - 4\lambda^2 - 7\lambda - 6 \\ &= -(\lambda+2)(\lambda^2 + 2\lambda + 3) \end{aligned}$$

with solutions $\lambda = -2$ and $\lambda = \frac{-2\pm\sqrt{-8}}{2} = -1 \pm i\sqrt{2}$.

We first find an eigenvector for $\lambda = -2$. In order to do this, we solve

$$0 = (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 0 \\ -2 & -1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -v_1 + 2v_3 \\ v_1 + v_2 \\ -2v_1 - v_2 + 2v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We observe that $v_1 = -v_2$ and $v_3 = \frac{1}{2}v_1$. The third equation vanishes with these substitutions, so v_2 is free to choose Choosing $v_2 = -2$ yields the eigenvector $\vec{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Next we consider the complex eigenvalues. We will choose one and find the real and imaginary parts of $e^{\lambda t}\vec{v}$. Choosing $\lambda = -1 + i\sqrt{2}$, we see that we need to solve

$$\begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -3v_1 + 2v_3 \\ v_1 - v_2 \\ -2v_1 - v_2 \end{pmatrix} = \begin{pmatrix} (-1+i\sqrt{2})v_1 \\ (-1+i\sqrt{2})v_2 \\ (-1+i\sqrt{2})v_3 \end{pmatrix}$$

The second equation yields $v_1 = i\sqrt{2}v_2$. Then, $2v_3 = (2+i\sqrt{2})v_1 = (2+i\sqrt{2})(i\sqrt{2})v_2 = (-2+2i\sqrt{2})v_2$, so $v_3 = (-1+i\sqrt{2})v_2$. With these substitutions the third equation vanishes, so v_2 is free to choose, Choosing $v_2 = 1$, we obtain the eigenvector $\vec{v} = \begin{pmatrix} i\sqrt{2} \\ 1 \\ -1+i\sqrt{2} \end{pmatrix}$. Hence,

$$\begin{aligned} e^{\lambda t}\vec{v} &= e^{(-1+i\sqrt{2})t}\begin{pmatrix} i\sqrt{2} \\ 1 \\ -1+i\sqrt{2} \end{pmatrix} \\ &= e^{-t}(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t))\begin{pmatrix} i\sqrt{2} \\ 1 \\ -1+i\sqrt{2} \end{pmatrix} \\ &= e^{-t}\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) + i\sqrt{2}\cos(\sqrt{2}t) \\ \cos(\sqrt{2}t) + i\sin(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) - i\sin(\sqrt{2}t) + i\sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}. \end{aligned}$$

Taking real and imaginary parts, it follows that the general solution is given by

$$\vec{x}(t) = c_1 e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + e^{-t}\left(c_2\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + c_3\begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}\right).$$

Setting $t = 0$,

$$\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} = \vec{x}(0) = c_1\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \left(c_2\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix}\right).$$

So, we have $2c_1 + \sqrt{2}c_3 = 0$, $-2c_1 + c_2 = -1$ and $c_1 - c_2 + \sqrt{2}c_3 = -2$. We see that $\sqrt{2}c_3 = -2c_1$ and $-c_2 = 1 - 2c_1$, so the third equation yields $c_1 =$. By back substitution, we find $c_2 = 1$ and $c_3 = -\sqrt{2}$ so

$$\begin{aligned} \vec{x}(t) &= e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + e^{-t}\left(\begin{pmatrix} -\sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} - \sqrt{2}\begin{pmatrix} \sqrt{2}\cos(\sqrt{2}t) \\ \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}\right) \\ &= e^{-2t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + e^{-t}\begin{pmatrix} -2\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \\ \cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \\ -3\cos(\sqrt{2}t) \end{pmatrix}. \end{aligned}$$

Exercise 8.8 (3.9.9). *Determine all vectors $\vec{x}_0$ such that the solution of the initial-value problem*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}\vec{x}, \quad \vec{x}(0) = \vec{x}_0$$

is a periodic function of time.

Sol. Via the eigenmethod, periodic contributions to the solution only come from complex eigenvalues; the rest yield either exponential growth or decay. Thus, we want initial conditions that are in the span of the eigenvectors we obtain from the complex eigenvalues. So, we determine the eigenvalues! These solve the characteristic equation

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ 1 & -1 & -1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(-1-\lambda) + 2(1-\lambda) = (1-\lambda)(\lambda^2 - 1 + 2) = (1-\lambda)(\lambda^2+1),$$

which has solutions $\lambda = 1$ and $\lambda = \pm i$. Since we only care about the span of eigenvectors coming from the complex eigenvalues, we won't compute the eigenvectors associated to $\lambda = 1$ and just refer to them as $c_1\vec{v}_1$.

For the complex eigenvalues, we choose $\lambda = i$ and will compute the real and imaginary parts of $e^{\lambda t}\vec{v}$, for associated eigenvector $\vec{v}$. This vector solves

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1 - 2v_3 \\ v_2 \\ v_1 - v_2 - v_3 \end{pmatrix} = \begin{pmatrix} iv_1 \\ iv_2 \\ iv_3 \end{pmatrix}.$$

It follows that $v_2 = 0$ and $v_1 = (1+i)v_3$. With these substitutions, the first equation vanishes and we see that v_3 is free to choose. Choosing $v_3 = 1$ yields the eigenvector $\begin{pmatrix} 1+i \\ 0 \\ 1 \end{pmatrix}$, and so

$$\begin{aligned} e^{\lambda t}\vec{v} &= e^{it} \begin{pmatrix} 1+i \\ 0 \\ 1 \end{pmatrix} \\ &= (\cos(t) + i\sin(t)) \begin{pmatrix} 1+i \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(t) - \sin(t) + i\cos(t) + i\sin(t) \\ 0 \\ \cos(t) + i\sin(t) \end{pmatrix}. \end{aligned}$$

Taking real and imaginary parts, we find that the general solution is of the form

$$\vec{x}(t) = c_1 e^t \vec{v}_1 + c_2 \begin{pmatrix} \cos(t) - \sin(t) \\ 0 \\ \cos(t) \end{pmatrix} + c_3 \begin{pmatrix} \cos(t) + \sin(t) \\ 0 \\ \sin(t) \end{pmatrix}.$$

Any solution with $c_1 = 0$ will be periodic! When is this the case? Observe that

$$\vec{x}(0) = \vec{x}_0 = c_1\vec{v}_1 + c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

so $c_1 = 0$ precisely when $\vec{x}_0$ is in the linear span of $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Since $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, the linear span is equivalently all vectors of the form $a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Hence, any initial condition of the form $\vec{x}_0 = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$ for constants a and b will yield periodic solutions.

9. Homework 9

Exercise 9.1 (3.10.1). *Find the general solution of*

$$\frac{d\vec{x}}{dt} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{pmatrix} \vec{x}.$$

Sol. We make use of the eigenmethod, first searching for eigenvalues of

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

solving the characteristic equation

$$\begin{aligned} 0 = \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} -\lambda & -1 & 1 \\ 2 & -3-\lambda & 1 \\ 1 & -1 & -1-\lambda \end{vmatrix} \\ &= -\lambda((-3-\lambda)(-1-\lambda)+1) + (2(-1-\lambda)-1) + -2 - (-3-\lambda) \\ &= -\lambda(4+4\lambda+\lambda^2) + -3 - 2\lambda + -2 + 3 + \lambda \\ &= -\lambda^3 - 4\lambda^2 - 5\lambda - 2 \\ &= -(\lambda+2)(\lambda+1)^2. \end{aligned}$$

We see that the eigenvalues of $\mathbf{A}$ are $\lambda = -2$ and $\lambda = -1$ (with multiplicity 2).

First we consider $\lambda = -2$, and search for an eigenvector $\vec{v}_1$ satisfying

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v}_1 = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \vec{v}_1.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the above equation is equivalent to solving

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v}_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_1.$$

We conclude that the first component of $\vec{v}_1$ must be zero and the second and third components must be equal, so any eigenvector is a constant multiple of $\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Next, we consider $\lambda = -1$, and first search for an eigenvector $\vec{v}_2$ satisfying

$$\vec{0} = (\mathbf{A} + \mathbf{I})\vec{v}_2 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \vec{v}_2.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the above equation is equivalent to solving

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v}_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_2.$$

We conclude that the third component of $\vec{v}_2$ must be zero, and the first and second components must be equal. Hence, any eigenvector is a constant multiple of $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

This yields two linearly independent solutions, but we need a third. To find this, we seek a generalized eigenvector $\vec{v}_3$ for $\lambda = -1$ satisfying $(\mathbf{A} + \mathbf{I})^2 \vec{v}_3 = 0$, as then we have the simple form

$$\begin{aligned} e^{\mathbf{A}t}\vec{v}_3 = e^{-t} e^{(\mathbf{A}+\mathbf{I})t} \vec{v}_3 &= e^{-t} \left(\mathbf{I} + t(\mathbf{A} + \mathbf{I}) + \frac{t^2}{2!}(\mathbf{A} + \mathbf{I})^2 + \cdots \right) \vec{v}_3 \\ &= e^{-t} \left(\vec{v}_3 + t(\mathbf{A} + \mathbf{I})\vec{v}_3 + \frac{t^2}{2!}(\mathbf{A} + \mathbf{I})^2 \vec{v}_3 + \cdots \right) \\ &= e^{-t} \left(\vec{v}_3 + t(\mathbf{A} + \mathbf{I})\vec{v}_3 + \mathbf{0} \right) \\ &= e^{-t} \left(\vec{v}_3 + t(\mathbf{A} + \mathbf{I})\vec{v}_3 \right). \end{aligned}$$

Computing, we need

$$\vec{0} = (\mathbf{A} + \mathbf{I})^2 \vec{v}_3 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix}^2 \vec{v}_3 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \vec{v}_3.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so the above equation is equivalent to solving

$$\vec{0} = (\mathbf{A} + \mathbf{I})\vec{v}_3 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_3.$$

We conclude only that the first two components must be equal. Since we seek a vector that is linearly independent from $\vec{v}_2$, we set the third component equal to 1 and choose $\vec{v}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as our generalized eigenvector. Observe that

$$(\mathbf{A} + \mathbf{I})\vec{v}_3 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Hence, the general solution is given by

$$\begin{aligned} \vec{x}(t) &= c_1 e^{-2t}\vec{v}_1 + c_2 e^{-t}\vec{v}_2 + c_3 e^{-t} \left(\vec{v}_3 + t(\mathbf{A} + \mathbf{I})\vec{v}_3 \right) \\ &= c_1 e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t} \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) \\ &= c_1 e^{-2t} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + e^{-t} \left(c_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} t+1 \\ t+1 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Up to redefining constants, we can rewrite this solution as

$$\vec{x}(t) = c_1 e^{-2t}\begin{pmatrix}0\\1\\1\end{pmatrix} + e^{-t}\left(c_2\begin{pmatrix}1\\1\\0\end{pmatrix} + c_3\begin{pmatrix}t\\t\\1\end{pmatrix}\right).$$

Exercise 9.2 (3.10.15)**.** *Suppose that* $\mathbf{A}^2 = \alpha\mathbf{A}$*. Find* $e^{\mathbf{A}t}$*.*

Sol. We determine the powers of $\mathbf{A}$ recursively. Notice that

$$\begin{aligned}\mathbf{A}^2 &= \alpha\mathbf{A}\\ \mathbf{A}^3 = \mathbf{A}^2\mathbf{A} &= \alpha^2\mathbf{A}\\ \mathbf{A}^4 = \mathbf{A}^3\mathbf{A} = \alpha^2\mathbf{A}^2 &= \alpha^3\mathbf{A}\\ \mathbf{A}^5 = \mathbf{A}^4\mathbf{A} = \alpha^3\mathbf{A}^2 &= \alpha^4\mathbf{A}\\ &\vdots\\ \mathbf{A}^n = \mathbf{A}^{n-1}\mathbf{A} = \alpha^{n-2}\mathbf{A}^2 &= \alpha^{n-1}\mathbf{A}\\ &\vdots\end{aligned}$$

Hence, we see that

$$\begin{aligned}e^{\mathbf{A}t} &= \mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots + \frac{t^n}{n!}\mathbf{A}^n + \cdots\\ &= \mathbf{I} + t\mathbf{A} + \frac{\alpha t^2}{2!}\mathbf{A} + \cdots + \frac{\alpha^{n-1}t^n}{n!}\mathbf{A} + \cdots\\ &= \mathbf{I} + \frac{1}{\alpha}\left(\alpha t\mathbf{A} + \frac{\alpha^2t^2}{2!}\mathbf{A} + \cdots + \frac{\alpha^n t^n}{n!}\mathbf{A} + \cdots\right)\\ &= \mathbf{I} + \frac{1}{\alpha}\left(-1 + 1 + \alpha t + \frac{\alpha^2t^2}{2!} + \cdots + \frac{\alpha^n t^n}{n!} + \cdots\right)\mathbf{A}\\ &= \mathbf{I} + \frac{e^{\alpha t} - 1}{\alpha}A.\end{aligned}$$

Exercise 9.3 (3.10.17)**.** *Let*

$$\mathbf{A} = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

1. *Show that* $\mathbf{A}^2 = -\mathbf{I}$.
2. *Show that*

$$e^{\mathbf{A}t} = \begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix}.$$

Proof. The first item is a straightforward computation, and we see

$$\mathbf{A}^2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 - 1 & \end{pmatrix} = -\mathbf{I}.$$

For the second item we could solve the equation $\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$, but instead we exploit the symmetry from the first point. Since $\mathbf{A}^2 = -\mathbf{I}$, we also have $\mathbf{A}^3 = -\mathbf{A}$ and $\mathbf{A}^4 = -\mathbf{A}^2 = \mathbf{I}$. Continuing in this way, for any $k \geq 0$, we have

$$\mathbf{A}^{4k} = \mathbf{I},\quad \mathbf{A}^{4k+1} = \mathbf{A},\quad \mathbf{A}^{4k+2} = -\mathbf{I},\quad \mathbf{A}^{4k+3} = -\mathbf{A}.$$

Hence, we can write

$$\begin{aligned} e^{\mathbf{A}t} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\mathbf{A}^n &= \sum_{\substack{n=4k\\k\geq 0}} \frac{t^n}{n!}\mathbf{I} + \sum_{\substack{n=4k+1\\k\geq 0}} \frac{t^n}{n!}\mathbf{A} - \sum_{\substack{n=4k+2\\k\geq 0}} \frac{t^n}{n!}\mathbf{I} - \sum_{\substack{n=4k+3\\k\geq 0}} \frac{t^n}{n!}\mathbf{A} \\ &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!}\right)\mathbf{I} + \left(\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!}\right)\mathbf{A} \\ &= \cos(t)\mathbf{I} + \sin(t)\mathbf{A}, \end{aligned}$$

where we have identified the Taylor series expansions of $\cos t$ and $\sin t$. Hence

$$e^{\mathbf{A}t} = \cos(t)\mathbf{I} + \sin(t)\mathbf{A} = \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & \sin t \\ -\sin t & 0 \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

as desired. □

Exercise 9.4 (3.11.1)**.** *Compute* $e^{\mathbf{A}t}$ *for*

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{pmatrix}.$$

Sol. We determine a fundamental matrix solution $\mathbf{X}$ for the equation

$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}$$

and compute $\mathbf{X}(t)\mathbf{X}(0)^{-1}$. To do so, we make use of the eigenmethod, finding eigenvalues of $\mathbf{A}$ which satisfy

$$\begin{aligned} 0 = \det(\mathbf{A} - \lambda\mathbf{I}) &= \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda)((3-\lambda)(-1-\lambda) - 1) + (-1-\lambda+3) - (1+3(3-\lambda)) \\ &= (1-\lambda)(\lambda^2 - 2\lambda - 4) + 2 - \lambda - 10 + 3\lambda \\ &= -\lambda^3 + 3\lambda^2 + 4\lambda - 12 \\ &= -(\lambda - 3)(\lambda - 2)(\lambda + 2). \end{aligned}$$

Thus, the eigenvalues of $\mathbf{A}$ are $\lambda = 3$, $\lambda = 2$ and $\lambda = -2$. We seek associated eigenvectors.

First, we consider $\lambda = 3$, and search for an eigenvector $\vec{v}_1$ satisfying

$$\vec{0} = (\mathbf{A} - 3\mathbf{I})\vec{v}_1 = \begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{pmatrix}\vec{v}_1.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & -\frac{5}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the above equation is equivalent to

$$\vec{0} = (\mathbf{A} - 3\mathbf{I})\vec{v}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\vec{v}_1.$$

We conclude that the second and third components of $\vec{v}_1$ must be equal, and the first and third components must be opposites. Hence, any eigenvector for $\lambda = 3$ is a constant multiple of $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, and $\vec{x}_1(t) = \begin{pmatrix} -e^{3t} \\ e^{3t} \\ e^{3t} \end{pmatrix}$.

Next, we consider $\lambda = 2$ and search for an eigenvector $\vec{v}_2$ satisfying

$$\vec{0} = (\mathbf{A} - 2\mathbf{I})\vec{v}_2 = \begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{pmatrix} \vec{v}_2.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{pmatrix} \to= \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, the above equation is equivalent to

$$\vec{0} = (\mathbf{A} - 2\mathbf{I})\vec{v}_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{v}_2.$$

We conclude that the second component of $\vec{v}_2$ must vanish, and the first and third components must be opposites. Hence, any eigenvector for $\lambda = 2$ must be a constant multiple of $\vec{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, and $\vec{x}_2(t) = \begin{pmatrix} -e^{2t} \\ 0 \\ e^{2t} \end{pmatrix}$.

Finally, we consider $\lambda = -2$ and search for an eigenvector $\vec{v}_3$ satisfying

$$\vec{0} = (\mathbf{A} + 2\mathbf{I})\vec{v}_3 = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{pmatrix} \vec{v}_3.$$

Proceeding via Gaussian elimination, we find

$$\begin{pmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{16}{3} & \frac{4}{3} \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}.$$

We conclude that the third component of $\vec{v}_3$ is 4 times the first, and -4 times the second. Thus, any eigenvector for $\lambda = -2$ is a constant multiple of $\vec{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$, and $\vec{x}_3(t) = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \\ 4e^{-2t} \end{pmatrix}$. Thus,

$$\mathbf{X}(t) = \begin{pmatrix} -e^{3t} & -e^{2t} & e^{-2t} \\ e^{3t} & 0 & -e^{-2t} \\ e^{3t} & e^{2t} & 4e^{-2t} \end{pmatrix}.$$

Hence

$$\mathbf{X}(0) = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 4 \end{pmatrix},$$

and we can find the inverse by Gaussian elimination on the augmented matrix

$$\begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & 1 & 0 & 1 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 & \frac{1}{5} & 1 & \frac{1}{5} \\ 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{5} & 0 & \frac{1}{5} \end{pmatrix}.$$

Thus,

$$\mathbf{X}^{-1}(0) = \frac{1}{5} \begin{pmatrix} 1 & 5 & 1 \\ -5 & -5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and so

$$e^{\mathbf{A}t} = \mathbf{X}(t)\mathbf{X}^{-1}(0) = \frac{1}{5} \begin{pmatrix} -e^{3t} & -e^{2t} & e^{-2t} \\ e^{3t} & 0 & -e^{-2t} \\ e^{3t} & e^{2t} & 4e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 5 & 1 \\ -5 & -5 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$
$$= \frac{1}{5} \begin{pmatrix} -e^{3t} + 5e^{2t} + e^{-2t} & -5e^{3t} + 5e^{2t} & -e^{3t} + e^{-2t} \\ e^{3t} - e^{-2t} & 5e^{3t} & e^{3t} - e^{-2t} \\ e^{3t} - 5e^{2t} + 4e^{-2t} & 5e^{3t} - 5e^{2t} & e^{3t} + 4e^{-2t} \end{pmatrix}.$$

Exercise 9.5 (3.11.15)**.** *Let $\mathbf{X}(t)$ be a fundamental matrix solution of*

$$\frac{d\vec{x}}{dt} = \mathbf{A}\vec{x}.$$

Prove that the solution $\vec{x}(t)$ of the initial-value problem

$$\begin{cases} \frac{d\vec{x}}{dt} = \mathbf{A}\vec{x} \\ \vec{x}(t_0) = \vec{x}_0 \end{cases}$$

is $\vec{x}(t) = \mathbf{X}(t)\mathbf{X}(t_0)^{-1}$.

Proof. Given a fundamental matrix solution $\mathbf{X}(t)$, we need then only find the constant vector $\vec{c}$ such that $\vec{x}(t) = \mathbf{X}(t)\vec{c}$ satisfies the initial conditions. Setting $t = t_0$, we see that $\vec{x}_0 = \vec{x}(t_0) = \mathbf{X}(t_0)\vec{c}$, and so $\vec{c} = \mathbf{X}(t_0)^{-1}\vec{x}_0$. Hence, our particular solution is

$$\vec{x}(t) = \mathbf{X}(t)\vec{c} = \mathbf{X}(t)\mathbf{X}(t_0)^{-1}.$$

□

Another way to see this would simply be by differentiating the formula for $\vec{x}(t)$ to see that it satisfies the requisite ODE, and plugging in $t = t_0$ directly to see that the initial conditions match.