Homework Solutions

Name: Notes and Solutions

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1. Homework 1

Exercise 1.1 (1.2.3)**.** *Find the general solution of*

$$\frac{dy}{dt} + \frac{2t}{1+t^2}y = \frac{1}{1+t^2}.$$

Sol. To solve this first order inhomogeneous linear ODE, we need an integrating factor g. Multiplying both sides by g, we obtain

$$g\frac{dy}{dt} + g\frac{2t}{1+t^2}y = \frac{g}{1+t^2}$$

and see that g must satisfy $\frac{dg}{dt} = g\frac{2t}{1+t^2}$. Since $\int \frac{2t}{1+t^2}dt = \ln|1+t^2| + c$, we see that g satisfies $\ln|g| = \ln|1+t^2| + c$, and so one integrating factor is $g = 1+t^2$. We can then rewrite the above equation as

$$\begin{aligned}(1+t^2)\frac{dy}{dt} + (1+t^2)\frac{2t}{1+t^2}y &= (1+t^2)\frac{1}{1+t^2}\\ (1+t^2)\frac{dy}{dt} + 2ty &= 1\\ \frac{d}{dt}\left((1+t^2)y\right) &= 1.\end{aligned}$$

Finding the antiderivative of both sides and rearranging, we see that the general solution is

$$y(t) = \frac{t+c}{1+t^2}.$$

Exercise 1.2 (1.2.7)**.** *Find the general solution of*

$$\frac{dy}{dt} + \frac{t}{1+t^2}y = 1 - \frac{t^3}{1+t^4}y.$$

Sol. We first rearrange this first order inhomogeneous linear ODE, writing

$$\frac{dy}{dt} + \left(\frac{t}{1+t^2} + \frac{t^3}{1+t^4}\right)y = 1.$$

We must once more find an integrating factor g. Multiplying both sides by g, we obtain

$$g\frac{dy}{dt} + g\left(\frac{t}{1+t^2} + \frac{t^3}{1+t^4}\right)y = g$$

and see that g must satisfy $\frac{dg}{dt} = g\left(\frac{t}{1+t^2} + \frac{t^3}{1+t^4}\right)$. Since $\int\left(\frac{t}{1+t^2} + \frac{t^3}{1+t^4}\right)dt = \frac{1}{2}\ln|1+t^2| + \frac{1}{4}\ln|1+t^4| + c$, we see that g satisfies $\ln|g| = \frac{1}{2}\ln|1+t^2| + \frac{1}{4}\ln|1+t^4| + c$ and hence that

$$g(t) = e^{\frac{1}{2}\ln(1+t^2)+\frac{1}{4}\ln(1+t^4)} = (1+t^2)^{1/2}(1+t^4)^{1/4}$$

is one integrating factor. We can then rewrite the above equation as

$$\begin{aligned}
\left((1+t^2)^{1/2}(1+t^4)^{1/4}\right)\frac{dy}{dt} + \left((1+t^2)^{1/2}(1+t^4)^{1/4}\right)\left(\frac{t}{1+t^2}+\frac{t^3}{1+t^4}\right)y &= (1+t^2)^{1/2}(1+t^4)^{1/4} \\
\frac{d}{dt}\left(\left((1+t^2)^{1/2}(1+t^4)^{1/4}\right)y\right) &= (1+t^2)^{1/2}(1+t^4)^{1/4} \\
\left((1+t^2)^{1/2}(1+t^4)^{1/4}\right)y(t) &= \int \left((1+t^2)^{1/2}(1+t^4)^{1/4}\right)\ dt + c \\
y(t) &= \frac{\int \left((1+t^2)^{1/2}(1+t^4)^{1/4}\right)\ dt + c}{(1+t^2)^{1/2}(1+t^4)^{1/4}}.
\end{aligned}$$

Exercise 1.3 (1.2.9)**.** *Solve the initial value problem*

$$\frac{dy}{dt} + \sqrt{1+t^2}e^{-t}y = 0, \quad y(0) = 1.$$

Sol. We can first rewrite this first order homogeneous linear ODE as

$$\frac{1}{y}\frac{dy}{dt} = -\sqrt{1+t^2}e^{-t},$$

or

$$\frac{d}{dt}\left(\ln|y|\right) = -\sqrt{1+t^2}e^{-t},$$

whenever $y \neq 0$. Integrating both sides from 0 to t, we find

$$\begin{aligned}
\int_0^t \frac{d}{ds}\left(\ln|y(s)|\right)\ ds &= \int_0^t -\sqrt{1+s^2}e^{-s}\ ds \\
\ln|y(t)| - \ln|y(0)| &= \int_0^t -\sqrt{1+s^2}e^{-s}\ ds \\
\ln|y(t)| &= \int_0^t -\sqrt{1+s^2}e^{-s}\ ds.
\end{aligned}$$

Unfortunately, we are unable to evaluate the integral on the right hand side. It follows that

$$y(t) = \pm e^{-\int_0^t \sqrt{1+s^2}e^{-s}\ ds},$$

and since $y(0) = 1 > 0$, we have

$$y(t) = e^{-\int_0^t \sqrt{1+s^2}e^{-s}\ ds}.$$

Exercise 1.4 (1.2.13)**.** *Solve the initial value problem*

$$\frac{dy}{dt} + y = \frac{1}{1+t^2}, \quad y(1) = 2.$$

Sol. We must find an integrating factor for this first order linear inhomogeneous ODE. Multiplying both sides by g, we find

$$g\frac{dy}{dt} + gy = \frac{g}{1+t^2},$$

we see that g must satisfy $\frac{dg}{dt} = g$. We know that one such solution is $g = e^t$, so we choose this as our integrating factor. We can thus rewrite the above equation as

$$\begin{aligned}
e^t\frac{dy}{dt} + e^t y &= \frac{e^t}{1+t^2} \\
\frac{d}{dt}\left(e^t y(t)\right) &= \frac{e^t}{1+t^2}.
\end{aligned}$$

Since we can't easily find an antiderivative for the right hand side, we use a definite integral and find

$$\begin{aligned}\int_1^t \frac{d}{ds}\left(e^s y(s)\right)\, ds &= \int_1^t \frac{e^s}{1+s^2}\, ds \\ e^t y(t) - e y(1) &= \int_1^t \frac{e^s}{1+s^2}\, ds \\ e^t y(t) &= 2e + \int_1^t \frac{e^s}{1+s^2}\, ds \\ y(t) &= e^{-t}\left(2e + \int_1^t \frac{e^s}{1+s^2}\, ds\right).\end{aligned}$$

Exercise 1.5 (1.2.17)**.** *Find a continuous solution of the initial value problem*

$$y' + y = g(t), \quad y(0) = 0$$

where

$$g(t) = \begin{cases} 2 & \text{if } 0 \le t < 1, \\ 0 & \text{if } t > 1. \end{cases}$$

Sol. Notice that g is discontinuous; as a result, we cannot guarantee that our solution y will be continuously differentiable. Our plan is to find general solutions to the given ODE on the intervals $[0,1)$ and $(1,\infty)$, and then use the initial conditions and the continuity assumption on y to give us specific solutions.

First, let us assume that $0 \le t < 1$. Then, our ODE can be written as

$$y' + y = 2.$$

As we found above, an integrating factor for $y' + y$ is e^t. Hence, we rewrite our equation as

$$\begin{aligned}e^t y' + e^t y &= 2e^t \\ \frac{d}{dt}\left(e^t y\right) &= 2e^t \\ e^t y(t) &= 2e^t + c \\ y(t) &= 2 + ce^{-t}.\end{aligned}$$

Since $y(0) = 0$, we have $2 + c = 0$ and hence $c = -2$. Thus, for $0 \le t \le 1$,

$$y(t) = 2 - 2e^{-t}.$$

Next, we consider $t > 1$. Then, our ODE can be written as

$$y' + y = 0.$$

This first order homogeneous linear ODE can be solved by rewriting it as

$$\begin{aligned}\frac{1}{y}\frac{dy}{dt} &= -1 \\ \frac{d}{dt}\left(\ln|y|\right) &= -1 \\ \ln|y| &= -t + c \\ y(t) &= ke^{-t},\end{aligned}$$

with k to be determined. Now, we use the continuity assumption. For our solution to be continuous, we need $\lim_{t\uparrow 1} = \lim_{t\downarrow 1} y(t) = y(1)$. So, we need

$$\lim_{t\uparrow 1} y(t) = \lim_{t\uparrow 1} 2 - 2e^{-t} = 2 - \frac{2}{e} = y(1) = \lim_{t\downarrow 1} y(t) = \lim_{t\downarrow 1} ke^{-t} = \frac{k}{e}.$$

Solving this equation for k, we find that $k = 2(e-1)$ and hence

$$y(t) = 2(e-1)e^{-t}$$

for $t > 1$. Thus, our full continuous solution is

$$y(t) = \begin{cases} 2 - 2e^{-t} & \text{if } 0 \le t \le 1; \\ 2(e-1)e^{-t} & \text{if } t > 1. \end{cases}$$

Note that the book should've specified that the ODE is not satisfied at $t = 1$; our solution has a cusp.

Exercise 1.6 (1.2.21)**.** *Determine the behavior of all solutions of the following ODE as $t \downarrow 0$:*

$$\frac{dy}{dt} + \frac{1}{\sqrt{t}} y = e^{\frac{\sqrt{t}}{2}}.$$

Sol. For $t > 0$, we solve this first order linear inhomogeneous ODE by finding an integrating factor. Multiplying both sides by g, we have

$$g\frac{dy}{dt} + \frac{g}{\sqrt{t}} y = g e^{\frac{\sqrt{t}}{2}}$$

and see that $\frac{dg}{dt} = \frac{g}{\sqrt{t}}$. Hence, $\ln|g(t)| = 2\sqrt{t} + c$, and one such integrating factor is $g(t) = e^{2\sqrt{t}}$. Thus, our equation becomes

$$\begin{aligned} e^{2\sqrt{t}}\frac{dy}{dt} + \frac{e^{2\sqrt{t}}}{\sqrt{t}} y &= e^{2\sqrt{t}} e^{\frac{\sqrt{t}}{2}} \\ \frac{d}{dt}\left(e^{2\sqrt{t}} y(t)\right) &= e^{\frac{5}{2}\sqrt{t}} \\ e^{2\sqrt{t}} y(t) &= \int e^{\frac{5}{2}\sqrt{t}}\, dt + c \\ y(t) &= \frac{\int e^{\frac{5}{2}\sqrt{t}}\, dt + c}{e^{2\sqrt{t}}}, \end{aligned}$$

where c will be determined by an initial condition of some variety. Notice that since $e^{\frac{5}{2}\sqrt{t}}$ is continuous for all t, so is its antiderivative. Furthermore, $e^{2\sqrt{t}} \to 1$ as $t \downarrow 0$. Hence, no matter what c is, $\lim_{t\downarrow 0} y(t)$ exists for the above solution!

Exercise 1.7 (1.4.1)**.** *Find the general solution of*

$$(1+t^2)\frac{dy}{dt} = 1 + y^2.$$

Sol. Rewriting this separable ODE as

$$\frac{1}{1+y^2}\frac{dy}{dt} = \frac{1}{1+t^2},$$

we see that

$$\frac{d}{dt}(\arctan(y)) = \frac{1}{1+t^2}.$$

Finding an antiderivative of both sides, we have

$$\arctan(y) = \arctan(t) + c,$$

or

$$y(t) = \tan(\arctan(t) + c)$$

$$= \frac{\tan(\arctan(t)) + \tan(c)}{1 - \tan(c)\tan(\arctan(t))}$$
$$= \frac{t+k}{1-kt}$$

for arbitrary k.

Exercise 1.8 (1.4.3)**.** *Find the general solution of*

$$\frac{dy}{dt} = 1 - t + y^2 - ty^2.$$

Sol. We rewrite this separable ODE as

$$\begin{aligned}\frac{dy}{dt} &= (1+y^2)(1-t)\\ \frac{1}{1+y^2}\frac{dy}{dt} &= 1-t\\ \frac{d}{dt}(\arctan(y)) &= 1-t.\end{aligned}$$

Finding an antiderivative of both sides, we have

$$\arctan(y) = t - \frac{1}{2}t^2 + c$$

or

$$y(t) = \tan\left(t - \frac{1}{2}t^2 + c\right)$$

for arbitrary c.

Exercise 1.9 (1.4.9)**.** *Solve the initial value problem*

$$\frac{dy}{dt} = \frac{3t^2 + 4t + 2}{2(y-1)}, \quad y(0) = -1$$

and determine the interval of existence of the solution.

Sol. We rewrite this separable ODE as

$$\begin{aligned}2(y-1)\frac{dy}{dt} &= 3t^2 + 4t + 2\\ \frac{d}{dt}\left(y^2 - 2y\right) &= 3t^2 + 4t + 2.\end{aligned}$$

Finding an antiderivative of both sides, we have

$$y(t)^2 - 2y(t) = t^3 + 2t^2 + 2t + c.$$

Since $y(0) = -1$, we have

$$(-1)^2 - 2(-1) = 3 = 0 + c$$

and so y satisfies

$$y(t)^2 - 2y(t) = t^3 + 2t^2 + 2t + 3.$$

Adding one to both sides,

$$y(t)^2 - 2y(t) + 1 = t^3 + 2t^2 + 2t + 4 = (y(t)-1)^2$$

and so

$$y(t) - 1 = \pm\sqrt{t^3 + 2t^2 + 2t + 4}.$$

Since $y(0) = -1$, we have

$$y(t) - 1 = -\sqrt{t^3 + 2t^2 + 2t + 4}$$

and hence

$$y(t) = 1 - \sqrt{t^3 + 2t^2 + 2t + 4}.$$

This solution is valid forwards in time from zero for all t since $t^3 + 2t^2 + 2t + 4 > 0$ for all $t > 0$, and is valid backwards in time until this cubic becomes negative. When does that happen? First, notice that this function is always increasing; its derivative is $3t^2 + 4t + 2$, which can be factored as

$$3t^2 + 4t + 2 = 3\left(t + \frac{2}{3}\right)^2 + \frac{2}{3} > 0.$$

Hence, the cubic has a unique zero. Plugging in $t = -2$ (motivated by the rational roots test), we see that this is a zero of the cubic. Hence, the solution is valid up until this point, so the interval of existence is $-2 < t < \infty$. y is defined at $t = -2$, but it doesn't technically solve the ODE because at that point $y(-2) = 1$ and the ODE is undefined.

Exercise 1.10 (1.4.11)**.** *Solve the initial value problem*

$$\frac{dy}{dt} = k(a - y)(b - y), \quad y(0) = 0,\ a, b > 0$$

and determine the interval of existence of the solution.

Sol. First, when $k = 0$, $y \equiv 0$ is a solution for all $-\infty < t < \infty$. In what follows, we assume $k \neq 0$. Then, we rewrite the separable equation as

$$\frac{1}{(a - y)(b - y)}\frac{dy}{dt} = k.$$

Suppose first that $a \neq b$. Then, we can split the left hand side using a partial fraction decomposition:

$$\frac{1}{(a - y)(b - y)} = \frac{1}{a - b}\left(\frac{1}{b - y} - \frac{1}{a - y}\right),$$

so

$$\begin{aligned}\left(\frac{1}{b - y} - \frac{1}{a - y}\right)\frac{dy}{dt} &= (a - b)k \\ \frac{d}{dt}\left(\ln|a - y| - \ln|b - y|\right) &= (a - b)k \\ \ln\left|\frac{a - y}{b - y}\right| &= (a - b)kt + c \\ \frac{a - y}{b - y} &= Ke^{(a-b)kt},\end{aligned}$$

with K to be determined. Since $y(0) = 0$¡ we have $\frac{a-0}{b-0} = \frac{a}{b} = K$, so

$$\frac{a - y}{b - y} = \frac{a}{b}e^{(a-b)kt}.$$

Rearranging, we find

$$\begin{aligned}a - y(t) &= (b - y(t))\frac{a}{b}e^{(a-b)kt} \\ a - y(t) &= ae^{(a-b)kt} - y(t)\frac{a}{b}e^{(a-b)kt}\end{aligned}$$

$$
\begin{aligned}
a\left(1-e^{(a-b)kt}\right) &= y(t)\left(1-\frac{a}{b}e^{(a-b)kt}\right)\\
y(t) &= \frac{a\left(1-e^{(a-b)kt}\right)}{1-\frac{a}{b}e^{(a-b)kt}}\\
y(t) &= \frac{ab\left(1-e^{(a-b)kt}\right)}{b-ae^{(a-b)kt}}.
\end{aligned}
$$

Notice that this solution matches the one in the back of the textbook, since

$$
\begin{aligned}
\frac{ab\left(1-e^{(a-b)kt}\right)}{b-ae^{(a-b)kt}} &= \frac{ab\left(e^{(a-b)kt}-1\right)}{ae^{(a-b)kt}-b}\\
&= \frac{ab\left(1-e^{-(a-b)kt}\right)}{a-be^{-(a-b)kt}}\\
&= \frac{ab\left(1-e^{(b-a)kt}\right)}{a-be^{(b-a)kt}}.
\end{aligned}
$$

This function is well defined so long as the denominator is nonzero, i.e. when $a = be^{(b-a)kt}$ or $t = \frac{1}{k(b-a)}\ln\frac{a}{b}$. The interval of existence then depends on whether or not k is positive or negative.

Why is that? Suppose $k > 0$. Then, when $a > b$ or when $a < b$ (it doesn't matter), one can see that $\frac{1}{k(b-a)}\ln\frac{a}{b}$ is negative. Hence, starting from $t = 0$, we can run our solution backwards in time to $\frac{1}{k(b-a)}\ln\frac{a}{b}$ or forwards in time to ∞ with no problem, and the interval of solution is $\frac{1}{k(b-a)}\ln\frac{a}{b} < t < \infty$. However, if $k < 0$, then no matter if $a > b$ or $b > a$, one can see that $\frac{1}{k(b-a)}\ln\frac{a}{b}$ is positive. Hence, starting from $t = 0$, we can run our solution forwards in time to $\frac{1}{k(b-a)}\ln\frac{a}{b}$, or backwards in time to $-\infty$ with no problem. Thus, if $k < 0$, the interval of solution is actualy $-\infty < t < \frac{1}{k(b-a)}\ln\frac{a}{b}$.

Finally, we consider $a = b$. Then, we have

$$
\begin{aligned}
\frac{1}{(a-y)^2}\frac{dy}{dt} &= k\\
\frac{d}{dt}\left(\frac{1}{a-y}\right) &= k\\
\frac{1}{a-y} &= kt + c,
\end{aligned}
$$

with c to be determined. Since $y(0) = 0$, we have $c = \frac{1}{a}$. Thus,

$$
\begin{aligned}
\frac{1}{a-y(t)} &= kt + \frac{1}{a}\\
1 &= akt + 1 - kty(t) - \frac{1}{a}y(t)\\
\left(kt + \frac{1}{a}\right)y(t) &= akt\\
y(t) &= \frac{a^2kt}{1+akt}.
\end{aligned}
$$

This function is well defined so long as the denominator is nonzero, i.e. when $1 + akt = 0$ or $t = -1/ak$. The interval of existence again depends on whether or not k is positive or negative. If $k > 0$, then $-1/ak$ is negative. Hence, we can run our solution from $t = 0$ backwards in time to $-1/ak$ or forwards in time to ∞ without issue, and the interval of existence is $\left(-\frac{1}{ak}, \infty\right)$. However, if $k < 0$, then $-1/ak$ is positive. Then, we can run our solution from $t = 0$ forwards in time to $-1/ak$ or backwards in time to $-\infty$ without issue, and the interval of existence is $\left(-\infty, -\frac{1}{ak}\right)$.

Exercise 1.11 (1.4.13)**.** *Any differential equation of the form $\frac{dy}{dt} = f(y)$ is separable. Thus, we can solve all those first-order differential equations in which time does not appear explicitly. Now, suppose we have a* *differential equation of the form* $\frac{dy}{dt} = f\left(\frac{y}{t}\right)$*, such as, for example, the equation* $\frac{dy}{dt} = \sin\left(\frac{y}{t}\right)$*. Differential equations of this form are called homogeneous equations. Since the right-hand side only depends on the single variable* $\frac{y}{t}$*, it suggests itself to make the substitution* $\frac{y}{t} = v$ *or* $y = tv$*.*

1. *Show that this substitution replaces the equation* $\frac{dy}{dt} = f\left(\frac{y}{t}\right)$ *by the equivalent equation* $t\frac{dv}{dt} + v = f(v)$*, which is separable.*

2. *Find the general solution of the equation* $\frac{dy}{dt} = 2\left(\frac{y}{t}\right) + \left(\frac{y}{t}\right)^2$*.*

Sol.

1. Since $y = tv$, we can differentiate both sides in t and find via the product rule that

$$\frac{dy}{dt} = t\frac{dv}{dt} + \frac{dt}{dt}v = t\frac{dv}{dt} + v.$$

Since $\frac{dy}{dt} = f\left(\frac{y}{t}\right) = f(v)$, we have

$$f(v) = \frac{dy}{dt} = t\frac{dv}{dt} + v,$$

as desired.

2. Here, $f(v) = 2v + v^2$. Using the above substitution, we can transform the equation $\frac{dy}{dt} = 2\left(\frac{y}{t}\right) + \left(\frac{y}{t}\right)^2$ into the separable

$$2v + v^2 = f(v) = t\frac{dv}{dt} + v.$$

Rewriting this separable equation, we have (as long as $v + v^2 \neq 0$)

$$\begin{aligned} t\frac{dv}{dt} &= v + v^2 \\ \frac{1}{v+v^2}\frac{dv}{dt} &= \frac{1}{t} \\ \left(\frac{1}{v(v+1)}\right)\frac{dv}{dt} &= \frac{1}{t} \\ \left(\frac{1}{v} - \frac{1}{v+1}\right)\frac{dv}{dt} &= \frac{1}{t} \\ \frac{d}{dt}\left(\ln\left|\frac{v}{v+1}\right|\right) &= \frac{1}{t}. \end{aligned}$$

Integrating both sides and solving for v, we have

$$\begin{aligned} \ln\left|\frac{v}{v+1}\right| &= \ln|t| + c \\ \frac{v}{v+1} &= ke^{\ln|t|} \\ \frac{v+1}{v} &= \frac{1}{kt} \\ 1 + \frac{1}{v} &= \frac{1}{kt} \\ \frac{1}{v} &= \frac{1-kt}{kt} \\ v(t) &= \frac{kt}{1-kt} \end{aligned}$$

for arbitrary $k \neq 0$. Since $v = y/t$, we have

$$y(t) = \frac{kt^2}{1 - kt}.$$

We now return to $v + v^2 \neq 0$. If $v + v^2 = 0$, then $v = 0$ or $v = -1$. $v = 0$ is accomplished by $y = 0$, which can be obtained from the previous general solution by allowing k to be zero. If $v = -1$ then $y = -t$. If that is the case, then $v + v^2 = 0$ so $\frac{dv}{dt} = 0$ and $v \equiv -1$ since v was -1 to begin with. Thus, $y = -t$ is also a solution and the general solution is

$$y(t) = -t \quad \text{or} \quad y(t) = \frac{kt^2}{1 - kt}.$$

2. Homework 2

Exercise 2.1 (1.5.1)**.** *Prove that* $\frac{a-bp_0}{a-bp(t)}$ *is positive for* $t_0 < t < \infty$.

Proof. We assume $a - bp_0 \neq 0$.

Following the hint from the textbook, we recall that for the logistic model a, b and p satisfy

$$a(t-t_0) = \ln \frac{p(t)}{p_0}\left|\frac{a-bp_0}{a-bp(t)}\right|$$

for $t_0 < t < \infty$. Notice that the left hand side is always well defined; as such, the right hand side must be as well. If $a - bp_0 \neq 0$ then, this forces $a - bp(t) \neq 0$ as well; otherwise, the right hand side would not be well defined.

At time $t = t_0$, the sign of $a - bp(t)$ trivially agrees with that of $a - bp_0$ (since they are equal!) and so the ratio is positive. This ratio will only cease to be positive then in the case that $a - bp(t)$ changes signs. Since $p(t)$ is differentiable (and hence continuous), we have by the Intermediate Value Theorem that in order to do so, there must be some point t^* at which $a - bp(t^*) = 0$. However, as we discussed above, this is impossible! Thus, $a - bp(t)$ is always the same sign as $a - bp_0$ for $t > t_0$, and we conclude that the ratio $\frac{a-bp_0}{a-bp(t)}$ is always positive. □

Now, of course notice that when $a - bp_0 = 0$, we discussed in class that from the equation one can see that $a - bp(t) = 0$ for all time $t > t_0$. As such, this case doesn't exactly satisfy the conclusion of the problem, but is a case that can be understood entirely on its own.

Exercise 2.2 (1.5.2)**.**

1. *Choose* 3 *times* t_0, t_1 *and* t_2, *with* $t_1 - t_0 = t_2 - t_1$. *Show that*

$$p(t) = \frac{ap_0 e^{a(t-t_0)}}{a - bp_0 + bp_0 e^{a(t-t_0)}} = \frac{ap_0}{bp_0 + (a - bp_0)e^{-a(t-t_0)}} \tag{1}$$

determine a *and* b *uniquely in terms of* t_0, $p(t_0)$, t_1, $p(t_1)$, t_2 *and* $p(t_2)$.

2. *Show that the period of accelerated growth for the United States ended in April,* 1914.

3. *Let a population* $p(t)$ *grow according to the logistic law (1), and let* $\bar{t}$ *be the time at which half the limiting population is achieved. Show that*

$$p(t) = \frac{\frac{a}{b}}{1 + e^{-a(t-\bar{t})}}.$$

Sol.

1. This approach follows the one discussed in Ed's notes. For ease, we set $p_i := p(t_i)$, and let p_∞ denote the limiting population $\frac{a}{b}$. We also set $\Delta := t_1 - t_0 = t_2 - t_1$. We seek formulae for a and b in terms of the t_i and p_i.

   The main idea behind this approach is that, while $p(t)$ is not a formula that can be easily manipulated, $1/p(t)$ is! We thus seek to understand $1/p(t)$ in a convenient way, and hope that will lead us to a smart algebraic solution. First, let us rewrite equation (1) a bit. Dividing both the numerator and denominator by bp_0, we have

$$p(t) = \frac{\frac{a}{b}}{1 + \left(\frac{a}{b}\frac{1}{p_0} - 1\right)e^{-a(t-t_0)}} = \frac{p_\infty}{1 + \left(\frac{p_\infty}{p_0} - 1\right)e^{-a(t-t_0)}}. \tag{2}$$

   Since p_0, p_∞ do not depend on time, we can further simplify the above formula by writing

$$p(t) = \frac{p_\infty}{1 + ce^{-a(t-t_0)}}$$

with $c = \frac{p_\infty}{p_0} - 1$. Thus, we have

$$p_0 = \frac{p_\infty}{1 + ce^{-a(t_0 - t_0)}} = \frac{p_\infty}{1+c}, \quad p_1 = \frac{p_\infty}{1 + ce^{-a(t_1 - t_0)}} = \frac{p_\infty}{1 + ce^{-a\Delta}}, \quad p_2 = \frac{p_\infty}{1 + ce^{-a(t_2 - t_0)}} = \frac{p_\infty}{1 + ce^{-2a\Delta}}.$$

Setting $\mu = e^{-a\Delta}$ for notational convenience, we have

$$p_0 = \frac{p_\infty}{1+c}, \quad p_1 = \frac{p_\infty}{1+c\mu}, \quad p_2 = \frac{p_\infty}{1+c\mu^2}.$$

Whew! This looks complicated, but we haven't actually done much yet - all we've done is simplified our formula to isolate important structure. This structure becomes evident if we now take reciprocals:

$$\frac{p_\infty}{p_0} = 1 + c, \quad \frac{p_\infty}{p_1} = 1 + c\mu, \quad \frac{p_\infty}{p_2} = 1 + c\mu^2.$$

These three equations will allow us to isolate μ (and hence a!) entirely in terms of given quantities. Notice that

$$\frac{p_\infty}{p_2} - \frac{p_\infty}{p_1} = c\mu^2 - c\mu = \mu(c\mu - c) = \mu\left(\frac{p_\infty}{p_1} - \frac{p_\infty}{p_0}\right)$$

and so

$$\mu = \frac{\frac{p_\infty}{p_2} - \frac{p_\infty}{p_1}}{\frac{p_\infty}{p_1} - \frac{p_\infty}{p_0}} = \frac{\frac{1}{p_2} - \frac{1}{p_1}}{\frac{1}{p_1} - \frac{1}{p_0}}.$$

Recalling that $\mu = e^{-a\Delta}$, we can entirely determine a in terms of the t_i and p_i as

$$\boxed{a = \frac{1}{\Delta}\ln\frac{1}{\mu} = \frac{1}{t_1 - t_0}\ln\left(\frac{\frac{1}{p_1} - \frac{1}{p_0}}{\frac{1}{p_2} - \frac{1}{p_1}}\right).}$$

Now, how do we determine b? Knowing μ, we can return to our previous equation $\frac{p_\infty}{p_1} = 1 + c\mu$ (or $\frac{p_\infty}{p_2} = 1 + c\mu^2$, but this is easier). Recalling that $c = \frac{p_\infty}{p_0} - 1$, we have

$$\begin{aligned} \frac{p_\infty}{p_1} &= 1 + \left(\frac{p_\infty}{p_0} - 1\right)\mu \\ \frac{p_\infty}{p_1} &= 1 - \mu + \frac{p_\infty}{p_0}\mu \\ p_\infty &= \frac{1-\mu}{\frac{1}{p_1} - \frac{\mu}{p_0}}. \end{aligned}$$

Recalling the formula for μ and $p_\infty = \frac{a}{b}$, we find that

$$\boxed{b = \frac{a}{p_\infty} = \frac{1}{t_1 - t_0}\ln\left(\frac{\frac{1}{p_1} - \frac{1}{p_0}}{\frac{1}{p_2} - \frac{1}{p_1}}\right)\frac{\frac{1}{p_1} - \frac{1}{p_0}\left(\frac{\frac{1}{p_2} - \frac{1}{p_1}}{\frac{1}{p_1} - \frac{1}{p_0}}\right)}{1 - \frac{\frac{1}{p_2} - \frac{1}{p_1}}{\frac{1}{p_1} - \frac{1}{p_0}}}.}$$

Whew! It is possible that this may be simplified, but we do not pursue that here. Notice, however, that both a and b are determined entirely in terms of t_i and p_i.

2. As we discussed in recitation, the period of accelerated growth ends when $p = \frac{a}{2b}$. One way to see this is that the period of accelerated growth ends when the first derivative begins decreasing, i.e. the second derivative switches from positive to negative. Computing the second derivative from $\frac{dp}{dt} = ap - bp^2$, we have

$$\frac{d^2p}{dt^2} = a\frac{dp}{dt} - 2bp\frac{dp}{dt} = (a - 2bp)p(a - bp).$$

Notice that for $p < \frac{a}{2b}$ all of the terms are positive and hence so is the second derivative. At $p = \frac{a}{2b}$ one term switches to negative, and the sign of the second derivative becomes negative as well. So, when does this happen? Let's use equation (1) to determine a formula for this time t^* in terms of a, b, t_0 and p_0. We have

$$\begin{aligned}
\frac{a}{2b} &= \frac{ap_0}{bp_0 + (a - bp_0)e^{-a(t^* - t_0)}} \\
\frac{2b}{a} &= \frac{b}{a} + \frac{a - bp_0}{ap_0} e^{-a(t^* - t_0)} \\
bp_0 &= (a - bp_0)e^{-a(t^* - t_0)} \\
\ln \frac{a - bp_0}{bp_0} &= a(t^* - t_0) \\
t^* &= t_0 + \frac{1}{a} \ln \frac{a - bp_0}{bp_0}.
\end{aligned}$$

Using the numbers $t_0 = 1790$, $p_0 = 3,929,000$, $a = 0.03134$ and $b = 1.5887 \times 10^{-10}$ from page 32 of the textbook, we find that $t^* \approx 1914.316$, which means that the period of accelerated growth ended some time in April, 1914.

3. *Proof.* Set $t_0 = \bar{t}$, and then $p_0 = \frac{a}{2b}$. Substituting these values into equation (1), we find

$$p(t) = \frac{a\frac{a}{2b}}{b\frac{a}{2b} + \left(a - b\frac{a}{2b}\right) e^{-a(t-\bar{t})}} = \frac{\frac{a^2}{2b}}{\frac{a}{2} + \left(a - \frac{a}{2}\right) e^{-a(t-\bar{t})}} = \frac{\frac{a}{b}}{1 + e^{-a(t-\bar{t})}},$$

as desired. □

Exercise 2.3 (1.5.3)**.** *In* 1879 *and* 1881 *a number of yearling bass were seined in New Jersey, taken across the continent in tanks by train, and planted in San Francisco Bay. A total of only* 435 *Striped Bass survived the rigors of these two trips. Yet, in* 1899*, the commercial net catch alone was* 1,234,000 *pounds. Since the growth of this population was so fast, it is reasonable to assume that it obeyed the Malthusian law* $\frac{dp}{dt} = ap$*. Assuming that the average weight of a bass fish is three pounds, and that in* 1899 *ever tenth bass fish was caught, find a lower bound for* a*.*

Sol. There is a question of when the surviving bass made the trip, but a lower bound for population growth would assume that they are all from the 1879 trip (this would be the situation of slowest possible population growth). Using the data about catch and weight in 1899, we can see that the population in 1899 was given by

$$1,234,000 \div 3 \times 10 = 4113333.\overline{3}.$$

Using the solution for Malthusian growth with $t_0 = 1879$ and $t = 1899$, we have

$$4113333.\overline{3} = p(1899) = p_0 e^{a(1899-1879)} = 435e^{20a}.$$

Unraveling and solving for a yields ≈ 0.458, which is a lower bound for the growth rate.

Exercise 2.4 (1.5.8)**.** *The population of New York City would satisfy the logistic law*

$$\frac{dp}{dt} = \frac{1}{25}p - \frac{1}{25 \times 10^6}p^2,$$

where t *is measured in years, if we neglected the high emigration and homicide rates.*

1. *Modify this equation to take into account the fact that* 9,000 *people per year move from the city, and* 1,000 *people per year are murdered.*

2. *Assume that the population of New York City was* 8,000,000 *in* 1970*. Find the population for all future time. What happens as* $t \to \infty$*?*

Sol.

1. We need to take account for that fact that each year, in addition to the population change, 10,000 people leave the model (independently of p). This can be accounted for by simply adding $-10,000$ to $\frac{dp}{dt}$:
$$\frac{dp}{dt} = \frac{1}{25}p - \frac{1}{25 \times 10^6}p^2 - 10,000.$$
2. For general constants this equation does not necessarily admit an exact solution, but in this case we can factor the right hand side and see that this is separable:
$$\begin{aligned} \frac{dp}{dt} &= \frac{1}{25}p - \frac{1}{25 \times 10^6}p^2 - 10,000 \\ &= -\frac{1}{25 \times 10^6}\left(p^2 - 10^6 p + 25 \times 10^{10}\right) \\ &= -\frac{1}{25 \times 10^6}\left(p - 5 \times 10^5\right)^2. \end{aligned}$$
Using $t_0 = 1970$ and $p_0 = 8 \times 10^6$, we have
$$\begin{aligned} \frac{-1}{(p - 5 \times 10^5)^2}\frac{dp}{dt} &= \frac{1}{25 \times 10^6} \\ \frac{d}{dt}\left(\frac{1}{p - 5 \times 10^5}\right) &= \frac{1}{25 \times 10^6} \\ \frac{1}{p(t) - 5 \times 10^5} - \frac{1}{8 \times 10^6 - 5 \times 10^5} &= \frac{1}{25 \times 10^6}(t - 1970) \\ p(t) = 5 \times 10^5 &+ \frac{1}{\frac{1}{25 \times 10^6}(t - 1970) + \frac{1}{8 \times 10^6 - 5 \times 10^5}}. \end{aligned}$$
Notice that as $t \to \infty$, the denominator of the fraction defining $p(t)$ tends to infinity. Thus, the fraction vanishes asymptotically and $p(t)$ approaches the limiting value of 500,000. This pessimistic model is clearly not very accurate!

Exercise 2.5 (1.9.3)**.** *Find the general solution of*
$$2t\sin y + y^3 e^t + (t^2\cos y + 3y^2 e^t)\frac{dy}{dt} = 0.$$

Sol. Notice that
$$\frac{\partial}{\partial y}\left(2t\sin y + y^3 e^t\right) = 2t\cos y + 3y^2 e^t = \frac{\partial}{\partial t}\left(t^2\cos y + 3y^2 e^t\right),$$
so that the equation is exact. Integrating $2t\sin y + y^3 e^t$ with respect to t, we find that
$$\Phi(t,y) = \int 2t\sin y + y^3 e^t \; dt = t^2\sin y + y^3 e^t + h(y)$$
for some function $h(y)$. Differentiating in y and matching to our above equation, we have
$$\frac{\partial}{\partial y}\left(t^2\sin y + y^3 e^t + h(y)\right) = t^2\cos y + 3y^2 e^t + h'(y) = t^2\cos y + 3y^2 e^t.$$
Thus $h'(y) = 0$, so h is just a constant and the general solution to our ODE is
$$\Phi(t,y) = t^2\sin y + y^3 e^t = \text{constant}.$$

Exercise 2.6 (1.9.7)**.** *Solve the initial value problem*
$$2ty^3 + 3t^2y^2\frac{dy}{dt} = 0, \quad y(1) = 1.$$

Sol. Notice that

$$\frac{\partial}{\partial y}\left(2ty^3\right) = 6ty^2 = \frac{\partial}{\partial t}\left(3t^2y^2\right)$$

so that the equation is exact. Integrating $2ty^3$ with respect to t, we find

$$\Phi(t,y) = \int 2ty^3 \ dt = t^2y^3 + h(y)$$

for some function $h(y)$. Differentiating in y and matching to our above equation, we have

$$\frac{\partial}{\partial y}\left(t^2y^3 + h(y)\right) = 3t^2y^2 + h'(y) = 3t^2y^2.$$

Thus, $h'(y) = 0$ and h is constant. The general solution to our ODE is then

$$\Phi(t,y) = t^2y^3 = \text{constant}.$$

Using $y(1) = 1$, we have $t^2y^3 = 1$ which allows us to solve for y as

$$y(t) = t^{-2/3}.$$

Exercise 2.7 (1.9.9)**.** *Solve the initial value problem*

$$3t^2 + 4ty + (2y + 2t^2)\frac{dy}{dt} = 0, \quad y(0) = 1.$$

Sol. Notice that

$$\frac{\partial}{\partial y}\left(3t^2 + 4ty\right) = 4t = \frac{\partial}{\partial t}\left(2y + t^2\right)$$

so that the equation is exact. Integrating $2y + 2t^2$ with respect to y, we find

$$\Phi(t,y) = \int (2y + 2t^2) \ dy = y^2 + 2t^2y + k(t)$$

for some function $k(t)$. Differentiating in t and matching to our above equation, we have

$$\frac{\partial}{\partial t}\left(y^2 + 2t^2y + k(t)\right) = 4ty + k'(t) = 3t^2 + 4ty,$$

so $k'(t) = 3t^2$ and $k(t) = t^3 + c$ for some c. The general solution to our ODE is then

$$\Phi(t,y) = y^2 + 2t^2y + t^3 = \text{constant}.$$

Since $y(0) = 1$, we actually have

$$y^2 + 2t^2y + t^3 = 1.$$

We can complete the square in the above as

$$(y + t^2)^2 = y^2 + 2t^2y + t^4 = 1 - t^3 + t^4.$$

The initial condition $y(0) = 1$ tells us to choose the positive square root, and hence

$$y(t) = -t^2 + \sqrt{1 - t^3 + t^4}.$$

Exercise 2.8 (1.9.13)**.** *Determine the constant a so that the following equation is exact, and then solve the resulting equation.*

$$\frac{1}{t^2} + \frac{1}{y^2} + \frac{(at+1)}{y^3}\frac{dy}{dt} = 0.$$

Sol. We require
$$\frac{\partial}{\partial y}\left(\frac{1}{t^2}+\frac{1}{y^2}\right)=\frac{-2}{y^3}=\frac{a}{y^3}=\frac{\partial}{\partial t}\left(\frac{at+1}{y^3}\right).$$
It is clear that $a=-2$ will make this equation exact. Integrating $\frac{1}{t^2}+\frac{1}{y^2}$ with respect to t yields
$$\Phi(t,y)=\int\left(\frac{1}{t^2}+\frac{1}{y^2}\right)\,dt=\frac{-1}{t}+\frac{t}{y^2}+h(y)$$
for some function $h(y)$. Differentiating in y and matching to our above equation, we have
$$\frac{\partial}{\partial y}\left(\frac{-1}{t}+\frac{t}{y^2}+h(y)\right)=\frac{-2t}{y^3}+h'(y)=\frac{-2t+1}{y^3}$$
so $h'(y)=\frac{1}{y^3}$ and $h(y)=\frac{-1}{2y^2}+c$. Hence,, the general solution to our ODE is
$$\Phi(t,y)=-\frac{1}{t}+\frac{t}{y^2}-\frac{1}{2y^2}=-\frac{1}{t}+\frac{2t-1}{2y^2}=k.$$
Rearranging, we have $\frac{2t-1}{2y^2}=k+\frac{1}{t}=\frac{kt+1}{t}$, so $y^2=\frac{t(2t-1)}{2(kt+1)}$. Taking square roots, we find
$$y(t)=\pm\sqrt{\frac{t(2t-1)}{2(kt+1)}}.$$

Exercise 2.9 (1.9.15)**.** *Show that every separable equation of the form* $M(t)+N(y)\frac{dy}{dt}=0$ *is exact.*

Proof. We need only notice that
$$\frac{\partial}{\partial y}(M(t))=0=\frac{\partial}{\partial t}(N(y)).$$
□

3. Homework 3

Exercise 3.1 (1.10.1). *Construct the Picard iterates for the initial-value problem*

$$\begin{cases} y' = 2t(y+1) \\ y(0) = 0 \end{cases}$$

and show that they converge to the solution $y(t) = e^{t^2} - 1$.

Proof. Recall that for $y'(t) = f(t, y)$ the Picard iterates satisfy the formula

$$y_n(t) = y_0 + \int_0^t f(s, y_{n-1}(s))\, ds,$$

which here is given by

$$y_n(t) = \int_0^t 2s(1 + y_{n-1}(s))\, ds.$$

We claim by induction that $y_n(t) = \sum_{k=0}^n \frac{t^{2k}}{k!} - 1$. The base case is clear: the previous would yield $y_0 = 1 - 1 = 0$, which is our initial condition. For the inductive step, we compute from the Picard iteration

$$\begin{aligned} y_{n+1}(t) &= \int_0^t 2s(1 + y_n(s))\, ds \\ &= \int_0^t 2s \sum_{k=0}^n \frac{s^{2k}}{k!}\, ds \\ &= \sum_{k=0}^n \int_0^t \frac{2s^{2k+1}}{k!}\, ds \\ &= \sum_{k=0}^n \frac{2t^{2k+2}}{(2k+2)k!} \\ &= \sum_{k=0}^n \frac{t^{2(k+1)}}{(k+1)!} = \sum_{k=0}^{n+1} \frac{t^{2k}}{k!} - 1, \end{aligned}$$

which is the desired formula. Hence, we conclude that $y_n(t) = \sum_{k=0}^n \frac{t^{2k}}{k!} - 1$ by induction.

To demonstrate the requested convergence, we observe that the nth order Taylor polynomials for $e^x - 1$ evaluated at $x = t^2$ are simply the y_n above. Hence, using the Lagrange form of the error, for every t there is some s with $0 \le s \le t^2$ such that

$$\begin{aligned} |y_n(t) - y(t)| &= \frac{d^{n+1}}{dx^{n+1}}(e^x - 1)\Big|_{x=s} \frac{s^{n+1}}{(n+1)!} \\ &= e^s \frac{s^{n+1}}{(n+1)!}. \end{aligned}$$

We observe that for any real number a, $\frac{a^n}{n!} \to 0$ (this is because eventually $n > a$, and so the successive terms are obtained by multiplication by a factor that vanishes asymptotically). Then, since s is independent of n, we see that $|y_n(t) - y(t)| = e^s \frac{s^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$, obtaining the desired convergence.

One could also bypass all of this work by just observing that the y_n are Taylor polynomials for e^x, and the Taylor series for e^x converges for all x. □

Exercise 3.2 (1.10.5). *Show that the solution* $y(t)$ *of the initial-value problem*

$$\begin{cases} y' = 1 + y + y^2 \cos(t) \\ y(0) = 0 \end{cases}$$

exists on the interval $0 \leq t \leq \frac{1}{3}$.

Sol. The goal is to apply Theorem 2. Notice that here we have a first order ODE of the form $y' = f(t, y)$, where $f(t, y) = 1 + y + y^2 \cos(t)$ and $\frac{\partial f}{\partial y} = 1 + 2y\cos(t)$ are continuous for all t and y. Letting a and b be arbitrary now, and letting M be the maximum value of $|f|$ on the rectangle $0 \leq t \leq a$, $|y - y_0| = |y| \leq b$, Theorem 2 guarantees existence up to time α, with

$$\alpha = \min\left(a, \frac{b}{M}\right).$$

We have no restrictions on our choices of a and b, since f and $\frac{\partial f}{\partial y}$ are continuous everywhere. Notice that f obtains its maximum by setting $t = 0$ and making y as large as possible on the rectangle. Hence, $M = 1 + b + b^2 \cos(0) = 1 + b + b^2$, and we can rewrite α in terms of a and b as

$$\alpha = \min\left(a, \frac{b}{1 + b + b^2}\right).$$

Notice that if we choose $a = b = 1$, then $\alpha = \frac{1}{3}$ and Theorem 2 guarantees the existence of the solution $y(t)$ on the interval $0 \leq t \leq \frac{1}{3}$.

Exercise 3.3 (1.10.7)**.** *Show that the solution* $y(t)$ *of the initial-value problem*

$$\begin{cases} y' = e^{-t^2} + y^2 \\ y(0) = 0 \end{cases}$$

exists on the interval $0 \leq t \leq \frac{1}{2}$.

Sol. The goal is to again apply Theorem 2. Notice that here we have a first order ODE of the form $y' = f(t, y)$, where $f(t, y) = e^{-t^2} + y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous for all t and y. Letting a and b be arbitrary now, and letting M be the maximum value of $|f|$ on the rectangle $0 \leq t \leq a$, $|y - y_0| = |y| \leq b$, Theorem 2 guarantees existence up to time α, with

$$\alpha = \min\left(a, \frac{b}{M}\right).$$

We have no restrictions on our choices of a and b, since f and $\frac{\partial f}{\partial y}$ are continuous everywhere. Notice that f obtains its maximum by setting $t = 0$ and making y as large as possible on the rectangle. Hence, $M = 1 + b^2$, and we can rewrite α in terms of a and b as

$$\alpha = \min\left(a, \frac{b}{1 + b^2}\right).$$

Notice that if we choose $a = b = 1$, then $\alpha = \frac{1}{2}$ and Theorem 2 guarantees the existence of the solution $y(t)$ on the interval $0 \leq t \leq \frac{1}{2}$.

Exercise 3.4 (1.10.17)**.** *Prove that* $y(t) = -1$ *is the only solution of the initial-value problem*

$$\begin{cases} y' = t(1 + y) \\ y(0) = -1 \end{cases}$$

.

Proof. Notice that $y(t) = -1$ solves the ODE since $y' \equiv 0$ and $t(y + 1) \equiv 0$. Thus, to prove that this is the only solution, we need only show that the solution to this ODE is unique.

One way to prove uniqueness is to apply Theorem 2' iteratively, tacking on a nonzero α of uniqueness time after each application, until uniqueness for all positive times has been demonstrated. That approach is a bit frustrating to write out rigorously however, so instead in these solutions I intend to mimic the proof of uniqueness in Theorem 2' in the textbook.

Suppose for the sake of a contradiction that there is a second solution $z(t)$ with the initial condition $z(0) = -1$. Then, since $z(t)$ solves the ODE it also solves the integral formulation

$$z(t) = -1 + \int_0^t s(1 + z(s))\ ds.$$

As a result, we can write

$$|y(t) - z(t)| = |-1 - z(t)| = |1 + z(t)| = \left| \int_0^t s(1 + z(s))\ ds \right| \leq \int_0^t s|1 + z(s)|\ ds$$

and see that

$$|1 + z(t)| \leq \int_0^t s|1 + z(s)|\ ds.$$

This is very similar to the setup of Lemma 2 in §1.10, and is an example of a more general inequality called **Grönwall's inequality**. I won't state that in full generality, but those who are interested should look it up! It's a very useful tool in the study of ODE and evolution type PDE.

What does this inequality give us? Set $w(t) = |1 + z(t)|$. Then, the above reads

$$w(t) \leq \int_0^t sw(s)\ ds,$$

where we take care to remark that w is nonnegative.

The heuristic is that we can think of $\int_0^t sw(s)\ ds = U(t)$ and for $t > 0$ write

$$\frac{1}{t}\frac{dU}{dt} \leq U(t).$$

Rewriting this, we have the inhomogeneous first order linear ODE

$$\frac{dU}{dt} - tU = f(t)$$

for some negative f, which we could solve by introducing the integrating factor $e^{-\frac{1}{2}t^2}$. Instead of dealing with the derivatives, we introduce this integrating factor directly into the integral inequality above.

For those of you who didn't want to read the above heuristic, we define

$$u(t) := e^{-\frac{1}{2}t^2} \int_0^t sw(s)\ ds.$$

Differentiating in t, we have by the product rule and the fundamental theorem of calculus that

$$u'(t) = -te^{-\frac{1}{2}t^2} \int_0^t sw(s)\ ds + e^{-\frac{1}{2}t^2} tw(t) = te^{-\frac{1}{2}t^2} \left(w(t) - \int_0^t sw(s)\ ds \right).$$

By the integral inequality $w(t) \leq \int_0^t sw(s)\ ds$, we can conclude that the right hand side is nonpositive for $t \geq 0$, i.e.

$$u'(t) \leq 0.$$

Hence, the function u is decreasing, and we have $u(t) \leq u(0) = 0$. Since u is positive for all $t \geq 0$, we have that $u(t) = 0$ for all $t \geq 0$. Then, $\int_0^t sw(s)\ ds$ is zero too, and since w is nonnegative we must have that $w(t) = |1 + z(t)| = 0$ for all time. Hence, $z(t) = -1 = y(t)$, and the solution to our ODE is unique! □

Exercise 3.5 (1.10.19). *Find a solution of the initial-value problem*

$$\begin{cases} y' = t\sqrt{1-y^2} \\ y(0) = 1 \end{cases}$$

other than $y(t) = 1$. *Does this violate Theorem 2'? Explain.*

Sol. Away from $y = 1$, we could solve this as a separable ODE, writing

$$\begin{aligned} \frac{1}{\sqrt{1-y^2}}\frac{dy}{dt} &= t \\ \frac{d}{dt}\left(\arcsin(y)\right) &= t \\ \arcsin(y(t)) &= \frac{1}{2}t^2 + c \\ y(t) &= \sin\left(\frac{1}{2}t^2 + c\right). \end{aligned}$$

Matching to our initial condition $y(0) = 1$, we have $1 = \sin(c)$ and so $c = \frac{\pi}{2}$. Thus, a second solution to our initial value problem is given by

$$y_2(t) = \sin\left(\frac{1}{2}t^2 + \frac{\pi}{2}\right),$$

and we see that our initial value problem has two solutions. This, however, does not violate the uniqueness guaranteed by Theorem 2' because Theorem 2' doesn't apply here. To apply Theorem 2', notice that $y' = f(t, y)$ with $f(t, y) = t\sqrt{1-y^2}$. We require f and $\frac{\partial f}{\partial y}$ to be continuous in a rectangle $0 \leq t \leq a$, $|y - y_0| = |y - 1| \leq b$ for some positive a and b. However, for any choice of $b > 0$, $y = 1$ is included in said rectangle and notice that

$$\frac{\partial f}{\partial y} = \frac{-ty}{\sqrt{1-y^2}}$$

is not continuous there! Hence, no such rectangle exists on which we can apply Theorem 2' and there is no reason to expect uniqueness.